Design of State Feedback Control Laws Based on Reduced Order Models

by

Fathi Abd Al-Adeem Abu Al-Saud

A Thesis Presented to the

FACULTY OF THE COLLEGE OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

In

ELECTRICAL ENGINEERING

June, 1990

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This thesis, written by ***FATHI ABD AL-ADEEM ABU AL-SAUD*** under the direction of his Thesis Advisor and approved by his Thesis Committee, has been presented to and accepted by the the Dean of the Collage of Graduate Studies, in partial fulfillment of the requirement of the degree of ***MASTER OF SCIENCE in ELECTRICAL ENGINEERING.***

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ACKNOWLEDGMENT

Acknowledgment is due to King Fahad University of Petroleum and Minerals for the support of this research.

I would like to express my appreciation to my thesis advisor Dr. Maamar Bettayeb for his patient guidance and encouragement throughout my thesis work.

I am also grateful to the other committee members, Dr. Samir Al-Baiyat, Dr. Ubaid Al-Saggaf, and Dr. Talal Bakri for their remarks.

Special thanks go to the current management of Consulting Services Department of Saudi Aramco for their unlimited support. I am also grateful to my colleague Ahmad Al-Shwaiheen for his help and encouragement

Also, I would like to express my deep appreciation to my wife for her patience and support.

Table of Contents

134

Table of Tables

Table

Table of Figures

Figure

خلاصة الرسالة

اسم الطالب: فتحي عبد العظيم ابو السعود

عنوان الرسالة: تصميم قوانين الضبط بواسطة التغذية المرتدة للحالات بناء على النماذج المبسطة

التخصص: هندسة كهربائية

تاريخ الرسالة: ذو القعدة ١٤١٠

سوف يشتق في هذا البحث قانون للضبط الخطي المربع الشبة مثالي للنظم ذات البعد الكبير بناء على النماذج المبسطة المشتقة بواسطة طريقة التوازن للتبسيط. و سوف تتم مقارنة القانون المقترح بالقوانين الشبة مثالية المبنية على النماذج المبسطة المشتقة بواسطة طريقة التجميع و طريقة التشويش الفردي. إن تقييم تأدية القوانين المختلفة على أساس الإستقرار و درجة الشبة مثالية وتدهور الثمن المربع و استجابات و ضوابط النظم المغلقة و مواقع القيم الفريدةللنظم المغلقة قد دل على تفوق القانون الشبة مثالي المقترح على القوانين الشبة مثالية الأخرى. لقد تمت المقارنة من خلال تطبيق هذه القوانين على معامل عده و كان أساس المقارنة هو القانون المثالي. وأيضا قد تم استنباط طريقة جديده للتبسيط مبنية على التوازن و التجميع.

كما تم استعراض إستخدام النماذج المبسطة المشتقة بواسطة طرق التوازن و التجميع و التشويش الفردي في اعادة وضع القيم الفريده. لقد تم إستعراض هدفين للتصميم: ١. اعادة وضع بعض القيم الفريده ٢. تشكيل استجابات النظام المفتوح إلى الأشكال المثالية. إن طريقة التجميع هي الأفضل بالنسبة للهدف الأول بينما لم تستطع أية طريقة كانت تحقيق الهدف الثاني.

درجة الماجستير في العلوم
جامعة الملك فهد للبترول و المعادن
الظهران المملكة العربية السعودية

ذو القعدة ١٤١٠

134

THESIS ABSTRACT

Author's Name: **FATHI ABD AL-ADEEM ABU AL-SAUD**

Title: **Design of State Feedback Control laws Based on Reduced Order Models**

Field: **Electrical Engineering**

Date : **June, 1990**

Suboptimal linear quadratic control law of high dimensional systems based on reduced order models derived via balancing reduction method is derived. The proposed law is compared to suboptimal control laws based on aggregation and singular perturbation. Performance evaluation based on stability, suboptimality degree, quadratic cost degradation, closed loop step responses and controls, and closed loop pole positions has shown the superiority of the proposed suboptimal control law. The comparison is done through the application of the suboptimal control laws to several physical plants. The base of the comparison is the optimal control law. Also, a new reduction method, which combines features of balancing and aggregation, called Balagg is introduced and evaluated.

The use of reduced order models based on balancing, aggregation, balagg, and singular perturbation in pole placement is also considered. In pole placement, two design purposes are considered: 1. placing only some of the open loop eigenvalues and 2. shaping the open loop responses to nominal ones. Aggregation and balagg perform the best for the first design purpose while none of reduction methods meet the second design purpose.

Throughout the thesis, the work is done for both continuous time and discrete time systems.

MASTER OF SCIENCE DEGREE

KING FAHAD UNIVERSITY OF PETROLEUM AND MINERALS
Dhahran, Saudi Arabia

June, 1990

CHAPTER 1

INTRODUCTION

1.1 General

Most practical systems we face in engineering today are complex in nature. Some examples are nuclear reactors, space crafts, power systems, chemical reactors, and distillation columns. Very often, the models of these systems are of large dimensions. Analysis, simulation, and design based on these high order models may eventually lead to complicated control strategies requiring very complex logic or large amounts of computation. Therefore, the use of model reduction becomes important in order to simplify the design of the control scheme and the simulation of the system.

Due to the nature of the mathematical models involved in describing large scale systems, many model reduction methods have been proposed. Most of the proposed reduction methods have a common plan. First, the elements of the model are separated into two parts according to the significance of their contributions to the desired performance characteristics of the system such as stability, optimality, and frequency response. Second, the most significant part of the model is used to design the control law for the overall system. Finally, the control law is implemented and system performance is tested through an extensive simulation.

Model reduction is concerned with obtaining a reduced order model $S2$ which will approximate the large scale system $S1$. $S1$ and $S2$ are given by

$$S1: \quad \dot{x}(t) = Ax(t) + Bu(t) \ , \quad x(0) = x_0 \qquad (1.1.1a)$$

$$y(t) = Hx(t) \qquad (1.1.1b)$$

where $x(t) \in R^n$, $u(t) \in R^m$, and $y(t) \in R^p$ are respectively the state vector , the input vector and the output vector at time t. The matrices A, B, and H are of dimensions nxn, nxm, and pxn respectively.

$$S2: \quad \dot{x}_1(t) = A_r x_1(t) + B_r u(t) \qquad (1.1.2a)$$

$$\hat{y}(t) = H_r x_1(t) \qquad (1.1.2b)$$

where $x_1(t) \in R^r$, and $\hat{y}(t) \in R^p$ are respectively the reduced order state vector and output vector at time t. The matrices A_r, B_r, and H_r are of dimensions rxr, rxm, and pxr respectively where $r < n$.

1.2 Literature Review

There have been an extensive literature on model reduction in control theory during the last three decades. There are time domain and frequency domain reduction methods. The most popular time domain reduction methods are aggregation, singular and regular perturbation, and balancing. Also, there are different versions of model reduction based on aggregation. These are exact, modal, chained, and continued fraction aggregation reduction methods. Among the frequency domain reduction methods are: Pade' approximation, Routh approximation, Pade-Routh, and moment matching.

In this thesis, we are going to be concerned with the time domain reduction methods only. The following subsections give a literature review of the time domain reduction methods for both open and closed loop systems.

1.2.1 Open Loop

Aggregation has long been a technique for analyzing static economic models. It has been a subject of extensive research studies in the economics literature [1-2]. In control theory, it has received a lot of attention as a very powerful reduction method which can be used for designing reduced order controllers. The first version of aggregation appeared in control theory is modal aggregation. It was first considered by Nicholson [3], Marshall [4], Davison [5-6] and Chidambara

[7-8]. The difference among their results is in the steady state response. While following different approaches, Marshall and Chidambara were able to obtain identical reduced order models which gives correct initial and steady state responses. However, the different models derived by Davison and Nicholson give rise to steady state error. It is interesting to note that the different modal aggregation methods result in exactly the same reduced order system matrix. The difference among them lies in the associated input and output matrices which in turn results in different responses. Having different input and output matrices is due to the different assumptions used in every reduction method.

Modal aggregation was further studied by many others. Lamba and Raw in [9] rederived Davison's model in a compact mathematical form and derived an aggregation matrix for it. Siret et. al in [10] gave a general formula for the aggregation matrix in terms of the inverse of the modal matrix of the full order system. Also, they showed that an aggregation matrix for Chidambara's model does not always exist.

Exact aggregation is the most general aggregation method. All other aggregation methods such as modal aggregation can be considered as special cases of exact aggregation. It was introduced by Aoki [11-12]. He established the notion of the aggregation matrix and derived the necessary conditions for a reduced order model to be considerd as an aggregated system of the full order one.

Chained aggregation is also a special case of exact aggregation. It is one of the more recent approaches in aggregation. It was developed by Tse et. al [13] then it was studied by Kwong [14]. It can be interpreted as a string of transformation which will transform the original full order system into a Generalized Hessenberg Representation (GHR).

Singular perturbation reduction method is a applicable to a certain class of large scale systems. The eigenvalues of the full order system can be divided into two categories: slow and fast. The slow subsystem of the full order system is then used as the reduced order system. There has been an extensive study on singular perturbation reduction method. Milne [15] provided conditions to check for weak coupling between subsystems. He referred to this type of systems as regularly perturbed systems. Kokotovic' and others contributed allot to singular perturbation reduction method, e.g. [16-22]. Slow subsystem and fast subsystem were derived and it was shown that the slow subsystem approximate the full order system. For the discrete time case, Mahmoud [23] also obtained the slow and fast subsystems and showed that the full order system can be approximated by the slow subsystem.

134

Balancing is the most recent reduction method. It is based on the idea that there exists a space coordinate system in which the controllability and the observability grammians are equal and diagonal. The reduced order model is then taken to be the most controllable and observable part of the full order system. Balancing was first introduced by Moore [24]. Mullis and Roberts [25], while synthesizing minimum round off noise fixed point digital filters, obtained a system very similar to a balanced system. Pernebo and Silverman [26] studied model reduction of systems based on balanced realization as well as the stability of the reduced order models. Laub and et.al [27] proposed a numerical method for obtaining the transformation matrix. Bettayeb and Silverman [28-29] further studied balancing. Bettayeb [30] studied discrete balancing. Bettayeb and Djennoune in [31] obtained a bound for the closeness of the eigenvalues of the subsystems and those of the full order system. Also, Al-Saggaf studied discrete balancing [32] and obtained a relation between the eigenvalues of the subsystems and the full order system for the discrete time case [33]. Fernando and Nicholson [34-36] further studied balancing and proposed using the slow subsystem of the balanced system as the reduced order model. This is particularly useful in discrete time systems because the reduced order system will also be balanced which is not the case if the usual reduction method is followed. However, Al-Saggaf and Franklin [37] proposed a method for obtaining a balanced reduced order system, not necessarily the slow subsystem, for discrete time systems. Sometimes, it is desirable to make the reduced order model has small error in a certain frequency band; therefor frequency weighted balancing has been introduced in [37] and L^∞ norm error bound has also been given. Enns [38] also considered the frequency weighted balancing which is based on a geometrical interpretation of balanced realization; however, he did not give L^∞ norm error bound. Verriest and Kailath [39] developed the theory of balanced realization of linear time varying systems. Also Shokoohi and Silverman [40] studied the balanced realization of linear time varying discrete time systems. Therapos [41] developed a method for balancing transformation of unstable nonminimal linear systems. Kenny and Hewer in [42] gave a necessary and sufficient condition for balancing unstable systems. Balanced realization of singularly perturbed systems have been studied by Shahruz and Behtash in [43] and by Bettayeb and Djennoune in [44]. Also, Liu and Anderson [45] obtained a reduced order system after approximating the balanced system by singularly perturbed form.

1.2.2 Closed Loop

If the dimension of the large scale system is high, then obtaining a controller for the system usually requires a lot of computation. Also the implementation of the controller might require a lot of hardware which may not be feasible. Although the amount of computation required might not be a problem nowadays because of the availability of fast computers and good numerical algorithms, the implementation of complicated control strategy has always been a problem.

To avoid the implementation of complicated controllers, reduced order controllers are used. There are two different approaches that have been followed to achieve controller reduction. These are:

1. Control Law Reduction Approach: a controller based on the high order system is first developed and then is reduced to a smaller one. This approach was studied by Wilson et. al [46]. Also, it was analyzed for balanced systems by Jonckheere and Silverman [47], Verriest [48], Yousuff and Skelton [49], Davis and Skelton [50], Glover [51], and Liu and Anderson [52]. Enns [38] used the frequency weighted balanced realization technique to reduce the high order controller. In [52], the methods proposed in [49-51] and [38], were applied to an eighth order plant in which the controller was reduced to different orders. It was shown that non of the methods always guarantees stability. Al-Saggaf [53] while working with the frequency weighted balanced realization technique, derived a procedure to reduce the high order controller which will result, under certain conditions, in a stable closed loop system.

2. Model Reduction Approach: a reduced order model is first obtained from the high order system by any model reduction method. Then a reduced order controller is obtained based on the reduced order model.

Reduced order models can be used into many areas in the control design of linear dynamical systems. Two of them are:

1. Linear Quadratic Regulator (LQR) control design.
2. Eigenvalue placement.

In aggregation, a suboptimal LQR control was proposed by Aoki [11] for exact aggregation. Lamba and Rao [9] developed an aggregation matrix for Davison's model and used the analysis of [11] to obtain a suboptimal LQR controller. Rao and Lamba [54] also derived a suboptimal LQR control strategy for models obtained via Chidambara reduction method. Also, Rao and Lamba [55] and Hickin and Sinha [56] proposed a method to reassign only r eigenvalues of the full order system using reduced order models.

In singular perturbation reduction method, Sannuti and Kokotovic' in [57] derived a suboptimal control law for regularly perturbed systems, and Chow and Kokotovic' in [58] based the suboptimal LQR control law for singularly perturbed system on the slow subsystem. Also in [58], a method is proposed to reassign the slow and the fast eigenvalues. Mahmoud [59]

134

also derived expressions for LQR suboptimal controller and eigenvalue placement for discrete time singularly perturbed systems.

In balancing, to the best of my knowledge, there have been no work on the design of suboptimal LQR controllers or eigenvalue placement based on reduced order models derived by balancing.

1.3 Problem Statement

The purpose of the thesis is to derive a suboptimal LQR control law based on reduced order models derived by balancing reduction method. Also, partial pole placement based on reduced order models of balanced systems will be evaluated. The proposed suboptimal LQR control law based on balancing will be compared to those based on aggregation and singular perturbation reduction methods. Also, suboptimal controls based on a new reduction scheme taking both features of balancing and aggregation are derived and evaluated. The comparison will be done through the reduction of practical systems to different reduced order models using the different reduction methods and then obtaining suboptimal controllers for every reduced order model. The basis of the comparison will be the optimal LQR controller based on the full order system. In the comparison, we are going to look at important performance characteristics such as stability, suboptimality, performance degradation, the system response to step input, and the controls. Also, a comparison will be done for partial pole placement.

1.4 Thesis Organization

The thesis will be divided into 6 chapters. Chapter 1 will give the introduction and state the purpose of the thesis. Chapter 2 will give a comprehensive coverage for the most popular continuous and discrete time reduction methods which are aggregation, singular perturbation, and balancing. In chapter 3 , suboptimal LQR controllers based on aggregation and singular perturbation will be given for both continuous and discrete time systems. Also, partial pole placement based on reduced order models obtained via aggregation and singular perturbation will be given in chapter 3.

In chapter 4, the proposed suboptimal LQR based on reduced order models via balancing will be derived. Also, the new reduction method based on balancing and aggregation will be presented. The simulation of practical examples and the comparison of the different suboptimal control laws based on the different reduction methods will be given for both continuous and discrete time systems. In chapter 5, the analysis of chapter 4 will be repeated for pole placement. Chapter 6 will give the conclusion.

134

CHAPTER 2

MODEL REDUCTION METHODS

2.1 Introduction

The model Simplification problem has received a great deal of attention during the last decade. Simplified models are derived for many purposes among which are:

1. Achieving simpler simulation of the process.
2. Reducing the computational effort and the complexity of designing optimal controllers based on the full order system by deriving suboptimal strategies based on the reduced order models.

This chapter gives a brief review of the open loop time domain model reduction methods: aggregation, singular perturbation, and balancing. Both continuous and discrete time cases will be reviewed.

The chapter is divided as follows. Section 2.2 gives an introduction to exact and modal aggregation reduction methods.

134

In section 2.3, singular perturbation reduction method will be reviewed. Balancing will be reviewed in section 2.4 and the summary is given in section 2.5.

2.2 Aggregation Methods

In control theory, aggregation has been of great interest because of the possibility it offers for providing simplified models which could be easily used in the analysis and synthesis of reduced order controllers. There are different approaches to achieve time domain aggregation among which are:

1. Exact aggregation
2. Modal (dominant pole) aggregation
3. Chained aggregation.

Exact aggregation was first introduced by Aoki [11-12]. Modal aggregation makes use of the modal matrix of the system. It was first considered by Nicholson [3], Marshall [4], Davison [5-6], and Chidambara [7-8]. Later it was further studied by many others [9-10]][54]. The main idea, in both exact and modal aggregation methods, is to find a reduced order model of order r which will preserve the dominant r eigenvalues of the full order system. Also , it is possible, by proper definition of the aggregation matrix, to let the reduced order model have any r eigenvalues , not necessarily the dominant ones, of the full order system.

Chained aggregation is one of the more recent approaches in aggregating a large scale linear time invariant systems. It was originally developed by Tse et al [13]. Later, it was further analyzed by Kwong [14]. Based on the unaggregated large scale system's information structure, the system is described through a "chain" of "aggregation" by a "Generalized Hessenberg Representation (GHR)". The procedure would allow one to obtain a reduced order model that preserves the input-output characteristics of the system behavior as manifested through the measurements (or outputs) available to him. It also allows one to discard the weakly observable modes of the system. Chained aggregation can be interpreted as a string of transformation which will transform the original full order system into the GHR form [13].

Since chained aggregation is not related to our analysis of closed loop behavior, it will not be further discussed. In the following subsections, an introductory coverage to exact and modal aggregation approaches will be given.

2.2.1 Exact Aggregation

Consider the controllable, observable, and stable large scale continuous linear time invariant system Sl described by:

$$Sl: \quad \dot{x}(t) = Ax(t) + Bu(t)\ , \quad x(0)=x_0 \qquad (2.2.1a)$$

$$y(t) = Hx(t) \qquad (2.2.1b)$$

where $x(t) \in R^n$, $u(t) \in R^m$, and $y(t) \in R^p$ are respectively the state vector , the input vector and the output vector at time t. The matrices A, B, and H are of dimensions nxn, nxm, and pxn respectively. It is desired to find an aggregated model $S2$ of dimension $r < n$ which can be considered as a reduced order model to $S1$.

$$S2: \quad \dot{x}_1(t) = Fx_1(t) + Gu(t) \qquad (2.2.2a)$$

$$\hat{y}(t) = Dx_1(t) \qquad (2.2.2b)$$

where $x_1(t) \in R^r$, and $\hat{y}(t) \in R^p$ are respectively the aggregated state vector , and the aggregated output vector at time t. The matrices F, G, and D are of dimensions rxr, rxm, and pxr respectively. $S2$ is obtained from $S1$ through the linear transformation [11]:

$$x_1(t) = Cx(t) \qquad (2.2.3)$$

where C is an rxn constant matrix of rank r called the aggregation matrix. Using (2.2.3), the equivalence between the models (2.2.1) and (2.2.2) is achieved provided that the following conditions are satisfied [11].

$$FC = CA \;;\; G = CB \;;\; DC = H \qquad (2.2.4a,b,c)$$

Since C is of full rank, it will posses a pseudo-inverse [61] and therefore a solution for (2.2.4a) is:

$$F = CAC'(CC')^{-1} \tag{2.2.5}$$

where (') indicates transpose. Due to the nature of the pseudo-inverse, it should be noted that the aggregated system matrix F obtained in (2.2.5) is an approximate solution to (2.2.4a).

If (2.2.4a) is satisfied, then it can be easily shown that F inherits r eigenvalues of A. To see this, let v_i be an eigenvector associated with λ_i where λ_i , $i=1,\ldots,n$, are the eigenvalues of A. It follows from (2.2.4a) that $CAv_i=\lambda_i Cv_i=FCv_i$. Therefore, if $Cv_i \neq 0$ then it is an eigenvector of F with an eigenvalue λ_i.

It is shown in [61] that a necessary and sufficient condition for (2.2.4a) to have a unique solution for F is that $Range(A'C') \subseteq Range(C')$. Also, it is shown in [72] that if $rank(C)=rank(CA)=r$ then the above condition is satisfied. It can be seen that this is always true if A is nonsingular.

There are basically two methods for determining the aggregation matrix C [11]. The first one is based on the knowledge of the eigenvalues and eigenvectors of A and also on the fact that F inherits r eigenvalues of A. The second method is based on the controllability matrices of $S1$ and $S2$.

Since the computation of the aggregated matrix F requires the calculation of the pseudo-inverse of the aggregation matrix C, then an error is expected. Define the aggregation error $e(t)$ as [11]:

$$e(t) = x_1(t) - Cx(t) \tag{2.2.6}$$

from (2.2.1), (2.2.2) and (2.2.6), the dynamics of the error are given by:

$$\dot{e}(t) = Fe(t) + (FC-CA)x(t) \tag{2.2.7}$$

If condition (2.2.4a) is satisfied then:

$$\dot{e}(t) = Fe(t) \tag{2.2.8}$$

Also if F is chosen to be stable, then $e(t)$ will asymptotically go to zero. In this case, the perfect aggregation is said to be obtained only asymptotically.

2.2.2 Modal Aggregation

All the methods that use the modal aggregation approach start with the derivation of the modal matrix M of the system matrix A. If the eigenvalues of A are distinct, which will be assumed throughout the thesis, then A can be transformed into purely diagonal form which is equal to $M^{-1}AM$. If the eigenvalues are repeated then A can be transformed into block diagonal form. There are basically two methods of modal aggregation. The first one was derived by Davison [5-6] and the second one by Chidambara [7-8] after many correspondences between them. The Chidambara's model corrects the steady state error that exist in Davison's model.

2.2.2.1 Continuous Time Analysis

A. Modal Aggregation Derived by Davison [5-6]

Although this method was originally developed by Davison, it was later represented by Lamba and Rao [9]. Because of the mathematical compactness of their representation, we are going to follow their approach in the following brief review of the method.

Again consider the systems $S1$ and $S2$ given by (2.2.1) and (2.2.2) respectively. Let

$$x(t) = Mz(t) \tag{2.2.9}$$

where M is the modal matrix of A with its columns arranged from left to right in the order of increasing magnitudes of the corresponding eigenvalues. From (2.2.1) and (2.2.9) we get

$$\dot{z} = \Lambda z + \Gamma u \tag{2.2.10}$$

where $$\Lambda = M^{-1}AM \;\; ; \;\; \Gamma = M^{-1}B \tag{2.2.11a,b}$$

Assuming that the first r dominant eigenvalues of A are to be retained in the simplified model, let

$$w = T_1 z \tag{2.2.12}$$

where $$T_1 = [\; I_r \;|\; 0 \;] \tag{2.2.13}$$

T_1 is of order rxn, I_r is the identity matrix of order r and w is an r-vector of the simplified model in modal form. It follows from (2.2.10), and (2.2.12) that the simplified system in modal form is given by

$$\dot{w} = T_1 \Lambda T_1^{+} w + T_1 \Gamma u \tag{2.2.14}$$

where T_1^{+} is the pseudo-inverse of T_1. T_1^{+}, in this case, equals to T_1'. Therefore, (2.2.14) can be written as

$$\dot{w} = T_1 \Lambda T_1' w + T_1 \Gamma u \tag{2.2.15}$$

In order to convert the modal representation of the simplified system into a general form, a reduced dimensional version of (2.2.9) is utilized, namely

$$x_1 = M_o w \tag{2.2.16}$$

The transformation matrix M_o is obtained in the following manner: Let the first r columns of the matrix M be represented by

$$\begin{bmatrix} v_1^1 & v_1^2 & \cdot & \cdot & \cdot & v_1^r \\ v_2^1 & v_2^2 & \cdot & \cdot & \cdot & v_2^r \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ v_n^1 & v_n^2 & \cdot & \cdot & \cdot & v_n^r \end{bmatrix} \tag{2.2.17}$$

Now from the first r columns of M, only the components that correspond to the r eigenvalues which are to be retained in the simplified model are left to form M_o, e.g. if on some physical grounds it is desired to keep state variables x_3, x_4 and x_n then

$$M_o = \begin{bmatrix} v_3^1 & v_3^2 & v_3^3 \\ v_4^1 & v_4^2 & v_4^3 \\ v_n^1 & v_n^2 & v_n^3 \end{bmatrix} \tag{2.2.18}$$

Consequently

$$\dot{x}_1 = Fx_1 + Gu \tag{2.2.19}$$

where $$F = M_o T_1 \Lambda T_1' M_o^{-1} \; ; \; G = M_o T_1 M^{-1} B \qquad (2.2.20a,b)$$

The r dimensional state vector x_1 is given by

$$x_1 = M_o w = M_o T_1 z = M_o T_1 M^{-1} x \qquad (2.2.21)$$

Thus the aggregation matrix C is given by

$$C = M_o T_1 M^{-1} \qquad (2.2.22)$$

Equation (2.2.19) gives the simplified model based on Davison, however, since the effect of the fast modes (non dominant eigenvalues) is completely neglected, steady state error occurs between the full order and the reduced order system responses. This was first pointed out through several correspondences between Davison and Chidambara [7]. In the next subsection, a model developed by Chidambara to resolve the steady state error will be presented.

B. Modal Aggregation Derived by Chidambara [7-8]

Chidambara, while using the dominant eigenvalue concept, developed a reduced order model in which the steady state (d.c) error , which exists in Davison's model, has been corrected [7]. In the following, the simplification technique due to Chidambara will be reviewed.

Partition the linear time invariant system Sl given in (2.2.1) as follows

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \tag{2.2.23}$$

The vector x_1 contains those r elements of the state vector that are to be retained in the simplified model. Doing similarity transformation as in (2.2.9), we can transform Sl to its modal representation given in (2.2.10). Assuming for simplicity that A has only distinct eigenvalues, and by arranging them in the order of dominance such that

$$|\lambda_1| < |\lambda_2| < \ldots < |\lambda_n| \tag{2.2.24}$$

and also by arranging the modal matrix column vectors respectively, then Λ can be written as

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \tag{2.2.25}$$

and (2.2.10) can be written as:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} u \tag{2.2.26}$$

where z_1 and z_2 are respectively $rx1$ and $(n-r)x1$ state vectors in modal form, $\Lambda_1 = diag(\lambda_1, \lambda_2, \ldots, \lambda_r)$, $\Lambda_2 = diag(\lambda_{r+1}, \lambda_{r+2}, \ldots, \lambda_n)$, Γ_1 and Γ_2 are respectively the top rxm and the bottom $(n-r)xm$ matrices of $M^{-1}B$. From (2.2.26)

$$\dot{z}_1 = \Lambda_1 z_1 + \Gamma_1 u \qquad (2.2.27a)$$

$$\dot{z}_2 = \Lambda_2 z_2 + \Gamma_2 u \qquad (2.2.27b)$$

Taking the Laplace transform of both sides of (2.2.27b) and ignoring the initial conditions gives

$$z_2(s) = (sI - \Lambda_2)^{-1}\Gamma_2 u(s) \qquad (2.2.28)$$

Ignoring all transient effects and only accounting for the d.c. transmission between $u(t)$ and $z_2(t)$ i.e. setting $s = 0$ in (2.2.28), we obtain the steady stste approximation

$$z_2(t) \approx -\Lambda_2^{-1}\Gamma_2 u(t) \qquad (2.2.29)$$

Now, equation (2.2.9) can be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \qquad (2.2.30)$$

so
$$x_1 = M_{11}z_1 + M_{12}z_2 \quad (2.2.31)$$
$$x_2 = M_{21}z_1 + M_{22}z_2 \quad (2.2.32)$$

(2.2.31) can be solved for z_1 in terms of x_1 and z_2 and then by substituting the value of z_1 in (2.2.32) and by making use of (2.2.29), we obtain a relation between x_1, and x_2 ,namely

$$x_2 = M_{21}M_{11}^{-1}x_1 + (M_{22}-M_{21}M_{11}^{-1}M_{12})(-\Lambda_2^{-1}\Gamma_2)u \quad (2.2.33)$$

Let
$$x_2 = Nx_1 + Lu \quad (2.2.34)$$

where
$$N = M_{21}M_{11}^{-1} \quad (2.2.35a)$$
$$L = (M_{22} - NM_{12})V \quad (2.2.35b)$$
$$V = -\Lambda_2^{-1}\Gamma_2 \quad (2.2.35c)$$

From (2.2.23),
$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1u \quad (2.2.36)$$

Substituting (2.2.34) into (2.2.36), the aggregated model is obtained as

$$\dot{x}_1 = Fx_1 + Gu \quad (2.2.37)$$

where
$$F = A_{11} + A_{12}N \;;\; G = B_1 + A_{12}L \quad (2.2.38a,b)$$

In the following section, a discrete version of modal aggregation will be reviewed.

2.2.2.2 Discrete Time Analysis

By following a parallel analysis to the continuous time case, a discrete time version of the aggregated system can be obtained for a discrete time full order system.

Consider the full order discrete system Sd given by:

$$Sd:\ x(k+1) = Ax(k) + Bu(k) \qquad (2.2.39a)$$

$$y(k) = Hx(k) \qquad (2.2.39b)$$

A discrete aggregated system Sdr of Sd is given by:

$$Sdr:\ x_1(k+1) = Fx_1(k) + Gu(k) \qquad (2.2.40a)$$

$$\hat{y}(k) = Dx_1(k) \qquad (2.2.40b)$$

For a discrete time version of Davison's model , the F and G matrices are given by (2.2.20 a,b).

Also, a discrete time version based on Chidambara model can be obtained. In this case, the matrices F and G of the aggregated system Sdr are as given by (2.2.38 a,b). However, since the steady state gain for discrete time systems is obtained by setting $z=1$ in the discrete transfer function, then V will be given by [23].

$$V = (I_{n-r} - \Lambda_2)^{-1}\Gamma_2 \qquad (2.2.58)$$

where I_{n-r} is the identity matrix of order $n-r$.

2.3 Singular Perturbation

Singular perturbation theory [16-17][20][66] provides an approach to deal with the modeling and analysis of dynamical systems in which the modes are separated into slow and fast and the separation is explicitly recognized in terms of a small, positive parameter multiplying a part of the system equations. In the following, a brief introduction to singular perturbation theory as an effective method for reducing large scale systems will be given. Since there is an extensive literature on singular perturbation, e.g. [16-17][20][66] only the results that are related to model reduction and related to our work will be reviewed.

2.3.1 Singularly Perturbed Continuous Time Linear Systems

The system dynamical equations in singularly perturbed form are given as [16-17][20]

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t); \quad x_1(0)=x_{10} \qquad (2.3.1a)$$

$$\mu\dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t); \quad x_2(0)=x_{20} \qquad (2.3.1b)$$

$$y(t) = C_1x_1(t) + C_2x_2(t) \qquad (2.3.1c)$$

where $x_1(t)$ and $x_2(t)$ are r and $n-r$ dimensional state subvector and μ is a positive scalar representing the ratio of speeds of slow and fast modes.

134

In applications, models of various physical systems can be put in form (2.3.1) by properly defining μ. For example, in power system models, μ can represent machine reactances or transients in voltage regulators [21], and in nuclear reactor models it is due to fast neutrons [22].

As shown in [63], the system in (2.3.1) possesses a two-time scale property, that is, it has r small eigenvalues of magnitude $O(1)$ [1] and $n-r$ large eigenvalues of magnitude $O(1/\mu)$. The system (2.3.1) can be considered as being composed of two subsystems: slow with small eigenvalues and fast with large eigenvalues [64]. In asymptotically stable system, the fast modes are important only during a short initial period. After that time, they are negligible and the behavior of the system can be described by the slow modes. Neglecting the fast modes is equivalent to assuming that they are infinitely fast that is letting $\mu \to 0$ in (2.3.1b). This is what is referred to as quasi steady state [64]. Setting μ to zero in (2.3.1b) gives

$$\dot{\bar{x}}_1(t) = A_{11}\bar{x}_1(t) + A_{12}\bar{x}_2(t) + B_1\bar{u}(t); \quad \bar{x}_1(0)=x_{10} \tag{2.3.2a}$$

$$0 = A_{21}\bar{x}_1(t) + A_{22}\bar{x}_2(t) + B_2\bar{u}(t) \tag{2.3.2b}$$

$$\bar{y}(t) = C_1\bar{x}_1(t) + C_2\bar{x}_2(t) \tag{2.3.2c}$$

---

1 A vector or matrix function $\Psi(\epsilon)$ of a positive scalar ϵ is said to be $O(\epsilon^k)$ if there exist positive constants c and ϵ^* such that $|\Psi(\epsilon)| \leq c\epsilon^k \quad \forall \epsilon \leq \epsilon^*$ [64]

where a bar indicates $\mu = 0$. Assuming that A_{22} is nonsingular, $\bar{x}_2$ can be expressed as

$$\bar{x}_2 = -A_{22}^{-1}(A_{21}\bar{x}_1 + B_2 u) \tag{2.3.3}$$

Substituting $\bar{x}_2$ into (2.3.1), the slow subsystem is defined as

$$\dot{x}_s = A_o x_s + B_o u_s \;; \quad x_s(0) = x_{10} \tag{2.3.4a}$$

$$y_s = C_o x_s + D_o u_s \tag{2.3.4b}$$

where

$$\begin{aligned} A_o &= A_{11} - A_{12}A_{22}^{-1}A_{21} \\ B_o &= B_1 - A_{12}A_{22}^{-1}B_2 \\ C_o &= C_1 - C_2 A_{22}^{-1}A_{21} \\ D_o &= - C_2 A_{22}^{-1}B_2 \end{aligned} \tag{2.3.4c}$$

Thus $\bar{x}_1 = x_s$, $\bar{y} = y_s$, $\bar{u} = u_s$ and $\bar{x}_2$ are the slow parts of the corresponding variables in (2.3.1). It is clear that letting $\mu = 0$ has reduced the order of the n dimensional system (2.3.1) to the r dimensional system (2.3.4). The slow model approximates the long term behavior of the full model system.

As mentioned earlier, the fast modes contributes to the system dynamical behavior only during a short initial period. To derive the fast subsystem, we assume that the slow variables are constant during fast transients, that is, $\dot{\bar{x}}_2=0$ and $\bar{x}_1=x_s$=constant. Using this in (2.3.1b) and (2.3.2), we obtain

$$\mu[\dot{x}_2 - \dot{\bar{x}}_2] = A_{22}[x_2 - \bar{x}_2] + B_2[u - u_s] \quad (2.3.5)$$

Defining

$$x_f = x_2 - \bar{x}_2 \;;\; u_f = u - u_s \;;\; y_f = y - y_s \quad (2.3.6)$$

then, the fast subsystem of (2.3.1) is defined as

$$\mu\dot{x}_f = A_{22}x_f + B_2u_f \;;\; x_f(0) = x_{20} - \bar{x}_2(0) \quad (2.3.7a)$$

$$y_f = C_2x_f \quad (2.3.7b)$$

It has been shown [64] that the reduced order model states x_s of (2.3.4) are $O(\mu)$ approximation to the unperturbed states of the system i.e. x_1, while to approximate the perturbed states of the system i.e x_2 then new states called "Boundary Layer Correction η " should be used together with the fast states. This correction is significant only during short period of time.

It has also been shown [18-19] that the structural properties of (2.3.1) (controllability, observability, and stability) can be deduced from those of the slow and fast subsystems.

2.3.2 Singularly Perturbed Discrete Time Linear Systems

There are different forms of discrete time singularly perturbed systems [66]. In the following, we are going to present only form D [66]. This is because of the its similarity with the continuous time singularly perturbed system.

Since the analysis of the discrete time case is very similar to that of the continuous time case, only the main differences will be highlighted.

Consider the discrete system given by [66]

$$x_1(k+1) = A_{11}x_1(k)+A_{12}x_2(k) + B_1u(k); x_1(0)=x_{10} \tag{2.3.8a}$$

$$\mu x_2(k+1) = A_{21}x_1(k)+A_{22}x_2(k) + B_2u(k); x_2(0)=x_{20} \tag{2.3.8b}$$

$$y(k) = C_1x_1(k) + C_2x_2(k) \tag{2.3.8c}$$

where $x_1(k)$ and $x_2(k)$ are r and $n-r$ dimensional state subvector and μ is a positive scalar representing the ratio of speeds of slow and fast modes.

To find the slow and the fast subsystems of the discrete time singularly perturbed system, we proceed as in the

134

continuous time case. The slow subsystem is found to be the discrete version of (2.3.4). Also the fast subsystem is found to be the discrete version of (2.3.7).

2.4 Balanced Reduction Method

Recently, an extensive study on balanced systems have been developed for the determination of a reduced order model which will contain only the most controllable and observable states of the system [24][26][28-37]. The main idea behind balancing is that there exist state space coordinates where the controllability and observability grammians are equal and diagonal. The diagonal entries of the grammians are called the singular values of the system and they provide a measure of how much a state is controllable and observable. A natural way, then, to achieve model reduction is to keep only the most controllable and observable states.

2.4.1 Continuous Time Analysis

In this section, we start with a review of the balanced reduction method then we will look at different characteristics of the balanced reduced order system.

2.4.1.1 Reduced Order System

Consider the controllable, observable and stable linear time invariant system $S1$ given in (2.2.1). The controllability and observability grammians (W_c and W_o respectively) of the system are given by

$$W_c = \int_0^\infty e^{At} BB' e^{A't} dt \qquad (2.4.1a)$$

$$W_o = \int_0^\infty e^{A't} H' H e^{At} dt \qquad (2.4.1b)$$

W_c and W_o are also the unique positive definite solutions of the following Lyapunov equations:

$$AW_c + W_c A' = -BB' \qquad (2.4.2a)$$

$$A'W_o + W_o A = -H'H \qquad (2.4.2b)$$

The grammians W_c and W_o are not invariant under equivalence transformations on the system Sl given in (2.2.1). Moore [24] showed that there exists a transformation T such that

$$W_c(T) = W_o(T) = \Sigma = diag(\sigma_1, \sigma_2, \ldots, \sigma_n) \qquad (2.4.3)$$

where $\qquad W_c(T) = TW_cT' \; ; \; W_o(T) = (T')^{-1} W_o T^{-1} \qquad (2.4.4a,b)$

and σ_i's are the singular values of the balanced system. The balanced system Sb is given by

$$Sb: \quad \dot{x}_b(t) = A_b x_b(t) + B_b u(t) \qquad (2.4.5a)$$

$$y(t) = C_b x_b(t) \qquad (2.4.5b)$$

where

$$A_b = TAT^{-1}; \; B_b = TB; \; C_b = HT^{-1}; \; x_b(t) = Tx(t) \quad (2.4.6)$$

Assume that the balanced states are ordered such that:

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0 \quad (2.4.7)$$

Let the balanced system be partitioned as

$$\begin{bmatrix} \dot{x}_{b1} \\ \dot{x}_{b2} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_{b1} \\ x_{b2} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \quad (2.4.8a)$$

$$y = [C_1 \quad C_2] \begin{bmatrix} x_{b1} \\ x_{b2} \end{bmatrix} \quad (2.4.8b)$$

where the vector $x_{b1} \in R^r$ contains the most controllable and observable states and the vector $x_{b2} \in R^{n-r}$ contains the least controllable and observable states. Also, let Σ be partitioned in a similar way

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \quad (2.4.9)$$

where $\Sigma_1 = diag(\sigma_1, \ldots, \sigma_r)$ and $\Sigma_2 = diag(\sigma_{r+1}, \ldots, \sigma_n)$. If $\sigma_r / \sigma_{r+1} \gg 1$, then the subsystem

$$\dot{x}_1 = A_{11}x_1 + B_1u \qquad (2.4.10a)$$

$$\hat{y} = C_1x_1 \qquad (2.4.10b)$$

is the reduced order model of the full order balanced system which will contain only the most controllable and observable part of the system [24][26][28-30].

2.4.1.2 Determination of the Transformation Matrix T

The following algorithm gives the steps necessary to find the transformation matrix T [27].

1. Solve the grammians W_c and W_o as the unique positive definite solutions of the Lyapunov equations (2.4.2).
2. Compute the Cholesky factors of the grammians

$$W_c = L_cL_c' \ ; \ W_o = L_oL_o' \qquad (2.4.11a,b)$$

where L_c and L_o denotes the lower triangular Cholesky factors of W_c and W_o respectively.
3. The singular value decomposition of the product of the Cholesky factors is computed as

$$L_o'L_c = U\Sigma V' \qquad (2.4.12)$$

4. The balanced transformation matrix T and its inverse are

then given by

$$T = \Sigma^{-1/2} U' L'_o \;;\; T^{-1} = L_c V \Sigma^{-1/2} \qquad (2.4.13a,b)$$

2.4.1.3 Properties of Balanced Systems

In the following, different properties of balanced systems and properties of the reduced order model will be reviewed.

Theorem 2.1 [26]

Assume that Σ_1 and Σ_2 have no diagonal entries in common. Then both subsystems (A_{ii}, B_i, C_i) $(i=1,2)$ of the balanced system, are asymptotically stable and internally balanced with grammians Σ_i $(i=1,2)$.

Theorem 2.2 [51]

An approximation error exists between the full order system and the reduced order model. An upper bound for this error is given by

$$\left\| C_b(j\omega I_n - A_b)^{-1} B_b - C_1(j\omega I_r - A_{11})^{-1} B_1 \right\|_\infty \leq 2(\sigma_{r+1} + \ldots + \sigma_n)$$
$$= 2tr(\Sigma_2) \qquad (2.4.14)$$

where tr denotes the trace and the infinity norm is defined as

$$|X(j\omega)|_\infty \triangleq \sup_{\omega \geq 0} = \bar{\sigma}[X(j\omega)] \qquad (2.4.15)$$

and $\bar{\sigma}(W)$ is the maximum singular value of W.

Theorem 2.3 [31]

Let the balanced system be partitioned as in (2.4.8). Assume that $\sigma_{r+1}/\sigma_r \ll 1$, then the system eigenvalues λ are contained in the regions defined as:

$$\left|(A_{ii} - \lambda I_{ni})^{-1}\right|^{-1} \le 2\left[\frac{\sigma_1 \sigma_{r+1}}{\sigma_r^2}\right]^{1/2} \left|A_{ii}^{-1}\right|^{-1}; \quad i=1,2 \qquad (2.4.16)$$

where n_1 and n_2 are the orders of the subsystems with $n_1 + n_2 = n$.

Corollary 2.1 [31]

If Σ_1 possesses a cluster of singular values σ_i, then (2.4.16) can be written as:

$$\left|(A_{ii} - \lambda I_{ni})^{-1}\right|^{-1} \le 2\left[\frac{\sigma_{r+1}}{\sigma_r}\right]^{1/2} \left|A_{ii}^{-1}\right|^{-1}; i=1,2 \qquad (2.4.17)$$

Define the separation term that characterizes the quality of the decomposition as

$$\beta = 2\left[\frac{\sigma_{r+1}}{\sigma_r}\right]^{1/2} \qquad (2.4.18)$$

If β is much less than one, then this result suggests close positions of the reduced order modes with respect to the original poles.

133

A property of SISO balanced systems is the symmetry property. One can show for SISO systems [34] that if the singular values are distinct, then the balanced system possesses the absolute value symmetry:

$$A_b = E A_b' E; \quad B_b = E C_b' \tag{2.4.19}$$

where E is a sign matrix defined as:

$$E = diag(e_i, i = 1, n); \quad e_i = \pm 1 \tag{2.4.20}$$

2.4.1.4 Singular Perturbation Approximation of Balanced System

Fernando and Nicholson in [35] proposed using the 'slow' subsystem of the balanced system Sb as the reduced order system. If the balanced system Sb is partitioned as in (2.4.8a), then

<u>Theorem 2.4 [35]</u>

If $\lambda_{max}(A_{11}) \ll \lambda_{min}(A_{22})$, then the slow subsystem of Sb as defined in (2.3.4) with D_o set to zero can be used as the reduced order system. Furthermore, the slow subsystem (A_o, B_o, C_o) of Sb is balanced with singular values Σ_1.

Using theorem 2.2, one can deduce [45] that even when the condition $\lambda_{max}(A_{11}) \ll \lambda_{min}(A_{22})$ does not hold, it is still possible to use the slow subsystem of the balanced system as the reduced order model as long as $2tr(\Sigma_2)$ is small.

We believe that it is more accurate to use the name 2-time scale approximation of balanced system instead of singular perturbation approximation of balanced system as used in [35] because Sb is not a singularly perturbed form. Note that the slow subsystem obtained using 2-time scale approximation is the same as that obtained using singular perturbation (see e.g [72]).

Balancing singularly perturbed systems has been recently proposed in [43] and in [44]. In [43], it is shown that it is possible to base the balancing transformation on the slow and fast subsystems individually. It is proved that the singular values of the singularly perturbed system are given by the union of the singular values of the slow and the fast subsystems with $O(\mu)$ approximation. Also, the transformation matrix T can be derived from the transformation matrices of the slow and fast subsystems which then will result in a balanced system which is $O(\mu)$ approximation to the balanced system derived based on the full order system.

2.4.2 Discrete Time Analysis

A review of the discrete time analogues of the results in section 2.4.1 will be given here. Although, the analysis of the discrete time case is very similar to that of the continuous time case, there are some major differences between the two.

2.4.2.1 Reduced Order System

Assume that the following discrete time system Sd is stable, controllable, and observable.

$$Sd:\ x(k+1) = Ax(k) + Bu(k) \qquad (2.4.21a)$$

$$y(k) = Hx(k) \qquad (2.4.21b)$$

Let the controllability and observability grammians W_c and W_o respectively be

$$W_c = \sum_{k=0}^{\infty} A^k BB'(A')^k \qquad (2.4.22a)$$

$$W_o = \sum_{k=0}^{\infty} (A')^k H'HA^k \qquad (2.4.22b)$$

W_c and W_o are also the unique positive definite solutions of the following Lyapunov equations:

$$AW_cA' - W_c = -BB' \qquad (2.4.23a)$$

$$A'W_oA - W_o = -H'H \qquad (2.4.23b)$$

133

As in the continuous time case, it is possible to find a transformation matrix which makes the two grammians diagonal and equal to Σ. The rest of the analysis required to obtain the reduced order system is completely analogues to that of the time case; therefore, it will not be repeated here. The discrete balanced system is (Φ, Γ, E) and the reduced order model is (Φ_1, Γ_1, E_1). There are some differences in the results between the continuous and discrete balanced cases and will be highlighted next.

<u>Theorem 2.5 [26]</u>

The subsystems $(\Phi_{ii}, \Gamma_{ii}, E_{ii})$, $(i = 1, 2)$ of the balanced system, are asymptotically stable. Also the subsystem (Φ_1, Γ_1, E_1) is controllable and observable but not internally balanced.

A procedure to obtain a balanced reduced order model for the discrete balanced system (Φ, Γ, E) is proposed in [37]. A bilinear transformation

$$s = \gamma \frac{z-\alpha}{z+\alpha} \Rightarrow z = \alpha \frac{\gamma+s}{\gamma-s} \qquad (2.4.24)$$

where $|\alpha| = 1$ and γ any positive real number which is not an eigenvalue of Φ, can be used to transform the discrete time system (Φ, Γ, E) to the continuous time system (A_b, B_b, C_b, D_b). If the discrete time system is balanced with grammian Σ, then the bilinear transformation will result in a balanced continuous time system with grammian Σ.

A balanced reduced order model (A_{11}, B_1, C_1, D) can then be obtained from the balanced system (A_b, B_b, C_b, D_b). Now, Using (2.4.24), (A_{11}, B_1, C_1, D) can then be transformed to the discrete time system $(\Phi_1, \Gamma_1, E_1, D_1)$ which, after some algebraic manipulation, can be expressed as

$$\Phi_1 = \Phi_{11} - \Phi_{12}(\alpha I + \Phi_{22})^{-1}\Phi_{21} \tag{2.4.25a}$$

$$\Gamma_1 = \Gamma_1 - \Phi_{12}(\alpha I + \Phi_{22})^{-1}\Gamma_2 \tag{2.4.25b}$$

$$E_1 = E_1 - E_2(\alpha I + \Phi_{22})^{-1}\Phi_{21} \tag{2.4.25c}$$

$$D_1 = - E_2(\alpha I + \Phi_{22})^{-1}\Gamma_2 \tag{2.4.25d}$$

<u>Theorem 2.6 [37]</u>

The reduced order model (Φ_1, Γ_1, E_1) is balanced with grammian Σ_1. Also, If Σ_1 and Σ_2 have no common entries then the reduced order model is asymptotically stable, observable and controllable.

Fernando and Nicholson in [36] obtained an equivalent result to (2.4.25) when $\alpha = -1$. Note that when $\alpha = -1$, then (2.4.25) represents the slow subsystem of the balanced system if it posses a 2-time scale property.

2.5 Summary

In this chapter, the most popular open loop reduction methods have been reviewed. The different versions of aggregation were presented. Also, the derivation of the slow and fast subsystems of a singularly perturbed system was reviewed. Model reduction based on aggregation and singular perturbation is based on the input-output characteristics of the system. The very recent model reduction method , Balancing, is based on the internal properties of the system which are the degree of controllability and observability of the states.

In the next chapter, the use of reduced order models obtained by aggregation and singular perturbation reduction methods in the design of suboptimal controllers and in pole placement will be reviewed. Suboptimal control and pole placement based on balancing will be presented in chapter 4 and chapter 5 respectively.

133

CHAPTER 3

CONTROL DESIGN USING SIMPLIFIED MODELS

3.1 Introduction

As mentioned previously, one of the main reasons for deriving reduced order models is to use them in designing simplified controllers. In the following, we are going to look at two applications of the reduced order models in the design of reduced order controllers. These are:

1. Suboptimal control using linear quadratic regulator control design.
2. Eigenvalue assignment using state feedback.

In section 3.2, the optimal control law based on the full order model will be reviewed. Then different suboptimal control laws based on the reduced order models obtained by the different aggregation methods presented earlier in section 2.2 will be reviewed. Also, a suboptimal control law based on the reduced order model of a singularly perturbed system will be reviewed. In section 3.3, discrete suboptimal control laws based on discrete aggregation and singular perturbation reduced order models will be studied. Using the reduced order models to

133

design control laws for assigning eigenvalues will be studied for aggregation and singular perturbation reduction methods in section 3.4. Finally, section 3.5 will give the summary.

Suboptimal control law and eigenvalue placement using reduced order models obtained by balancing will be presented in chapter 4.

3.2 Suboptimal LQR Control Design for Continuous Time Systems

The aim in suboptimal control is to find an optimal control law based on a reduced order model which, when implemented as a suboptimal policy to the full order system, will result in a closed loop system very close to the one that will be obtained when the control law is derived based on the full order system.

In order to evaluate the suboptimal control laws based on the reduced order models obtained by the different reduction methods presented earlier, we first review the optimal control law based on the full order system.

3.2.1 Optimal Control

Consider the controllable, observable, and stable large scale continuous linear time invariant system $S1$ described by:

$$S1: \quad \dot{x}(t) = Ax(t) + Bu(t) \;, \quad x(0) = x_0 \qquad (3.2.1a)$$

$$y(t) = Hx(t) \qquad (3.2.1b)$$

133

where $x(t) \in R^n, u(t) \in R^m$, and $y(t) \in R^p$ are respectively the state, the input and the output vectors at time t. The matrices A, B, and H are with dimensions nxn, nxm and pxn respectively. It is desired to minimize the cost functional J associated with (3.2.1) given as

$$J = \int_0^\infty (x'Qx + u'Ru)\, dt \tag{3.2.2}$$

where $Q \in R^{nxn}$ is a symmetric nonnegative definite matrix and $R \in R^{mxm}$ is a symmetric positive definite matrix. The optimal control law that minimizes (3.2.2) is given by [67]

$$u^* = -Kx \tag{3.2.3}$$

where the gain matrix K is

$$K = R^{-1}B'P \tag{3.2.4}$$

and P is the solution of the Ricatti equation

$$A'P + PA - PBR^{-1}B'P + Q = 0 \tag{3.2.5}$$

If the optimal control law obtained in (3.2.3) is applied to (3.2.1) then the optimal closed loop system is obtained as

133

$$S^*: \quad \dot{x} = (A - BK)x \tag{3.2.6}$$

Also the value of the cost functional J is [67]

$$J^* = x_0' P x_0 \tag{3.2.7}$$

If the dimension of $S1$ is large then the computation involved in solving for K will be very lengthy ($n(n+1)/2$ nonlinear algebraic equations must be solved to obtain P [67]). Also, the implementation of the control law (3.2.3) may require a lot of components which may not be feasible. Therefore, it is very desirable to obtain a lower order control law which will result in a reduction in the computation involved and in the complexity of implementing the controller. This reduced order controller will serve as a suboptimal controller for the system $S1$. In the following, a review of suboptimal control policies based on aggregation and singular perturbation reduction methods will be given.

3.2.2 Suboptimal Control Via Aggregation Methods

In the following, the reduced order models obtained by the different aggregation methods discussed in chapter 2 will be used to design a suboptimal control law for the full order system $S1$.

133

A. Exact Aggregation

The aggregated system of $S1$ as shown in chapter 2 is give as

$$S2: \quad \dot{x}_1(t) = Fx_1(t) + Gu(t) \tag{3.2.8a}$$

$$\hat{y}(t) = Dx_1(t) \tag{3.2.8b}$$

where $x_1(t) \in R^r$, and $\hat{y}(t) \in R^p$ are respectively the aggregated state vector , and the aggregated output vector at time t. The matrices F, G and D are with dimensions rxr, rxm and pxr respectively. Associate with the reduced order system $S2$ the following cost functional [11]

$$J_m = \int_0^\infty (x_1' Q_m x_1 + u'Ru)\ dt \tag{3.2.9}$$

where $Q_m \in R^{rxr}$ is a symmetric nonnegative definite matrix. The optimal control law that minimizes (3.2.9) is given by [67]

$$u = -K_a x_1 \tag{3.2.10}$$

where the gain matrix K_a is

$$K_a = R^{-1} G' P_a \tag{3.2.11}$$

and P_a is the solution of the Ricatti equation

$$F'P_a + P_aF - P_aGR^{-1}G'P_a + Q_m = 0 \qquad (3.2.12)$$

x_1 is related to x by the aggregation matrix C as

$$x_1(t) = Cx(t) \qquad (3.2.13)$$

where C is an rxn constant aggregation matrix of rank r. Using (3.2.13), the control law in (3.2.10) can be expressed as

$$u = -K_aCx \qquad (3.2.14)$$

If the control law obtained in (3.2.14) is applied to (3.2.1), then the suboptimal closed loop system is obtained as

$$\dot{x} = (A - BK_aC)x \qquad (3.2.15)$$

For $S2$ to be an aggregated system of $S1$, then the exact aggregation conditions [11]

$$FC = CA \ ; \ G = CB \ ; \ DC = H \qquad (3.2.16a,b,c)$$

should be satisfied. If C posses a pseudo-inverse [61], then (3.2.16a) can be solved for F as

$$F = CAC'(CC')^{-1} \qquad (3.2.17)$$

From (3.2.16) and (3.2.17) and premultiplying and postmultiplying (3.2.12) by C' and C respectively, we obtain

$$A'C'P_aC + C'P_aCA - C'P_aCBR^{-1}B'C'P_aC + C'Q_mC = 0 \quad (3.2.18)$$

Comparing (3.2.18) to (3.2.5), we can see that $C'P_aC$ corresponds to P if $C'Q_mC$ is made to correspond to Q. They can not be equated because P and Q are of rank n while $C'P_aC$ and $C'Q_mC$ are at the most of rank r. So, if Q_m is suitably chosen, then the control law given in (3.2.14) can serve as a suboptimal control law for SI. If Q_m is chosen as [11]

$$Q_m = (CC')^{-1}CQC'(CC')^{-1} \quad (3.2.19)$$

then, the suboptimal control law for the system is given as

$$u = -K_{sa}x \quad (3.2.20a)$$

where $$K_{sa} = K_aC \quad (3.2.20b)$$

B. Modal Aggregation

An aggregation matrix for Davison's model [5-6] was developed by Lamba and Rao [9]. The aggregation matrix C is given as

$$C = M_oT_1M^{-1} \quad (3.2.22)$$

where M is the system modal matrix, $T_1 = [I_r | 0]$, I_r is the identity matrix of order r, and M_o is chosen depending on what states are to be left in $S2$. In [9], the suboptimal control law is then obtained by following the same procedure of exact aggregation but using the value of C as given in (3.2.22).

Rao and Lamba [54] also developed a procedure to find the suboptimal control law based on Chidambara's model [7-8]. The following is a review of the procedure.

Consider the simplified system given in (3.2.8) where F and G are

$$F = A_{11} + A_{12}N \;\; ; \;\; G = B_1 + A_{12}L \qquad (3.2.23a,b)$$

and N and L are

$$N = M_{21}M_{11}^{-1} \qquad (3.2.24a)$$

$$L = (M_{22} - NM_{12})V \qquad (3.2.24b)$$

$$V = -\Lambda_2^{-1}\Gamma_2 \qquad (3.2.24c)$$

A_{ii}, B_i, M_{ii}, $i=1,2$ are the submatrices of A, B, and M respectively. $\Lambda_2 = diag[\lambda_{r+1},\dots,\lambda_n]$ and Γ_2 is the bottom $(n-r)$ rows of $M^{-1}B$. If the weighting matrix Q is partitioned as

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \tag{3.2.25}$$

where $Q_{21} = Q'_{12}$, then the cost functional J given in (3.2.2) can be written as

$$J = \int_0^\infty (x'_1 Q_{11} x_1 + 2x'_1 Q_{12} x_2 + x'_2 Q_{22} x_2 + u'Ru)dt \tag{3.2.26}$$

The simplified cost functional J_m associated with simplified model is obtained by substituting the value of x_2

$$x_2 = Nx_1 + Lu \tag{3.2.27}$$

into (3.2.26). After some algebraic simplification, J_m can be written as

$$J_m = \int_0^\infty (x'_1 Q_1 x_1 + 2x'_1 Su + u'R_1 u)dt \tag{3.2.28}$$

where

$$Q_1 = Q_{11} + 2N'Q'_{12} + N'Q_{22}N \tag{3.2.29a}$$

$$R_1 = R + L'Q_{22}L \tag{3.2.29b}$$

$$S = Q_{12}L + N'Q_{22}L \tag{3.2.29c}$$

Let

$$\tilde{u} = u + R_1^{-1}S'x_1 \tag{3.2.30}$$

Substituting (3.2.30) into (3.2.8a), a simplified model in terms of $\bar{u}$ is obtained as

$$\dot{x}_1 = (F - GR_1^{-1}S')x_1 + G\bar{u} \qquad (3.2.31)$$

Also, the simplified cost functional J_m becomes

$$J_m = \int_0^\infty (x_1' Q_m x_1 + \bar{u}' R_1 \bar{u})\, dt \qquad (3.2.32)$$

where

$$Q_m = Q_1 - SR_1^{-1}S' \qquad (3.2.33)$$

If Q_m is positive semi-definite and R_1 is positive definite then the optimal control law that minimizes (3.2.32) is given by [68]

$$\bar{u} = -R_1^{-1}G'P_a x_1 \qquad (3.2.34)$$

where P_a is the solution of the matrix Riccati equation

$$P_a(F-GR_1^{-1}S') + (F-GR_1^{-1}S')'P_a - P_aGR_1^{-1}G'P_a + Q_m = 0 \qquad (3.2.35)$$

From (3.2.30) and (3.2.34), the optimal controller for the simplified model (3.2.8) is

$$u = -K_a x_1 \tag{3.2.36a}$$

where
$$K_a = -R_1^{-1}(G'P_a + S') \tag{3.2.36b}$$

The control law in (3.2.36) can serve as a suboptimal controller to the full order model SI if only x_1 states are accessible. However, to find a suboptimal controller for SI in case that all the states are accessible then an aggregation matrix that relates x_1 to x must be available. In [10], it was shown that such an aggregation matrix for Chidambara's model does not always exist.

3.2.3 Suboptimal Control Via Singular Perturbation Method

In a singularly perturbed system, the suboptimal control law is based on the slow subsystem. This is due to the fact that the system is influenced by the slow subsystem much more than by the fast subsystem.

Consider the singularly perturbed system

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t) \;;\; x_1(0)=x_{10} \tag{3.2.37a}$$

$$\mu\dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t) \;;\; x_2(0)=x_{20} \tag{3.2.37b}$$

$$y(t) = C_1x_1(t) + C_2x_2(t) \tag{3.2.37c}$$

where $y \in R^p$ is the output vector, $x_1(t)$ and $x_2(t)$ are respectively r and $(n-r)$ dimensional state subvectors and μ is a positive scalar representing the ratio of speeds of slow and fast modes.

The slow subsystem is given as [63]

$$\dot{x}_s = A_o x_s + B_o u_s \; ; \quad x_s(0) = x_{10} \tag{3.2.38a}$$

$$y_s = C_o x_s + D_o u_s \tag{3.2.38b}$$

where

$$\begin{aligned} A_o &= A_{11} - A_{12}A_{22}^{-1}A_{21} \\ B_o &= B_1 - A_{12}A_{22}^{-1}B_2 \\ C_o &= C_1 - C_2A_{22}^{-1}A_{21} \\ D_o &= - C_2A_{22}^{-1}B_2 \end{aligned} \tag{3.2.38c}$$

$x_s \in R^r$, $y_s \in R^p$, and $u_s \in R^m$ are respectively the slow subsystem state, output and input vectors. It is desired to minimize the cost functional J_s associated with the slow subsystem [63].

$$J_s = \int_0^\infty (y_s' y_s + u_s' R u_s) dt \tag{3.2.39}$$

(3.2.39) can be written in terms of x_s and u_s as

$$J_s = \int_0^\infty (x_s' C_o' C_o x_s + 2u_s' D_o' C_o x_s + u_s' R_o u_s) dt \tag{3.2.40}$$

where R_o is given by

$$R_o = R + D_o' D_o \tag{3.2.41}$$

Note that (3.2.40) is similar to (3.2.28); therefore, by following a similar analysis to the analysis which led to the optimal control law given in (3.2.36), it is possible to find that the optimal control for (3.2.38) which minimizes (3.2.40) is given by

$$u_s^* = -K_{sp}x_s \qquad (3.2.42a)$$

where

$$K_{sp} = -R_o^{-1}(B_o'K_s + D_o'C_o) \qquad (3.2.42b)$$

where K_s is the positive semidefinite solution to the Ricatti equation

$$K_s(A_o - B_oR_o^{-1}D_o'C_o) + (A_o - B_oR_o^{-1}D_o'C_o)'K_s - K_sB_oR_o^{-1}B_o'K_s + C_o'(I_p - D_oR_o^{-1}D_o')C_o = 0 \qquad (3.2.43)$$

where I_p is the identity matrix of order p.

The following theorem gives a sufficient condition for the existence and uniqueness of K_s [63].

<u>Theorem 3.1</u>

If the triple (A_o, B_o, C_o) is stabilizable and detectable then (3.2.43) has a unique positive semidefinite stabilizing solution K_s.

133

To find the suboptimal control law for the full order system (3.2.37) then [63]

$$u = -K_{ss}x \tag{3.2.44a}$$

where

$$K_{ss} = [K_{sp} | 0] \tag{3.2.44b}$$

The suboptimal control law depends entirely on the slow part of the system only. The fast subsystem is completely ignored.

In the following section, suboptimal controls via aggregation and singular perturbation will be reviewed for the discrete time case.

3.3 Suboptimal LQR Control Design For Discrete Time Systems

The analysis of the discrete time case is very similar to that of the continuous time case. Therefore, only the main differences will be highlighted.

3.3.1 Discrete Time Optimal Control

Consider the controllable, observable, and stable large scale discrete linear shift invariant system Sd described by:

$$Sd: \quad x(k+1) = Ax(k) + Bu(k) \;, \; x(0) = x_0 \tag{3.3.1a}$$

$$y(k) = Hx(k) \tag{3.3.1b}$$

where $x(k) \in R^n$, $u(k) \in R^m$, and $y(k) \in R^p$ are respectively the state, the input and the output vectors at interval k. The matrices A, B, and H are with dimensions nxn, nxm, and pxn respectively. It is desired to minimize the cost functional J associated with (3.3.1) given as

$$J = \sum_{k=0}^{\infty} (x'(k)Qx(k) + u'(k)Ru(k)) \qquad (3.3.2)$$

where $Q \in R^{nxn}$ is a symmetric nonnegative definite matrix and $R \in R^{mxm}$ is a symmetric positive definite matrix. The optimal control law that minimizes (3.3.2) is given by [67]

$$u^* = -Kx \qquad (3.3.3)$$

where the gain matrix K is

$$K = (R + B'PB)^{-1}B'PA \qquad (3.3.4)$$

and P is the solution of the discrete Ricatti equation

$$P^{-1} - A'PA + A'PB(R + B'PB)BP'A - Q = 0 \qquad (3.3.5)$$

If the optimal control law obtained in (3.3.3) is applied to (3.3.1) then the optimal closed loop system is

133

$$S^*: \quad x(k+1) = (A - BK)x(k) \tag{3.3.6}$$

Also the value of the cost functional J is [67]

$$J^* = x_0' P x_0 \tag{3.3.7}$$

As in the continuous time case, If the dimension of Sd is large then the computation involved in solving for K will be very lengthy. Also, the implementation of the control law (3.3.3) may require a lot of components which may not be feasible. Therefore, it is very desirable to obtain a lower order control law based on a reduced order model. In the following, a review of suboptimal control policies based on aggregation and singular perturbation discrete reduction methods will be given.

3.3.2 Discrete Time Suboptimal Control Via Aggregation Methods

In the following, the reduced order models obtained by the different aggregation methods discussed in chapter 2 will be used to design a suboptimal control law for the full order system Sd.

A. Exact Aggregation

The discrete aggregated system of Sd as shown in chapter 2 is give as

$$Sdr: \quad x_1(k+1) = Fx_1(k) + Gu(k) \tag{3.3.8a}$$

$$\hat{y}(k) = Dx_1(k) \tag{3.3.8b}$$

where $x_1(k) \in R^r$, and $\hat{y}(k) \in R^p$ are respectively the aggregated state vector , and the aggregated output vector at interval k. The matrices F, G, and D are with dimensions rxr, rxm, and pxr respectively. Associate with the reduced order system Sdr the following cost functional [11]

$$J_m = \sum_{0}^{\infty} (x_1' Q_m x_1 + u'Ru) \tag{3.3.9}$$

where $Q_m \in R^{rxr}$ is a symmetric nonnegative definite matrix. The optimal control law that minimizes (3.3.9) is given by [67]

$$u = -K_a x_1 \tag{3.3.10}$$

where the gain matrix K_a is

$$K_a = (R + G'P_aG)^{-1}G'P_aF \tag{3.3.11}$$

and P_a is the solution of the discrete Ricatti equation

$$P_a^{-1} - F'P_aF + F'P_aG(R + G'P_aG)GP_a'F - Q_m = 0 \quad (3.3.12)$$

Since x_1 is related to x by the aggregation matrix C then the control law in (3.3.10) can be expressed in terms of x as

$$u = -K_{sa}x \quad (3.3.13a)$$

where

$$K_{sa} = K_aC \quad (3.3.13b)$$

If the control law obtained in (3.3.13) is applied to (3.3.1), then the suboptimal closed loop system is obtained as

$$x(k+1) = (A - BK_aC)x(k) \quad (3.3.14)$$

To find Q_m, then a parallel analysis to that of the continuous time case is followed. It can be shown that Q_m is given by

$$Q_m = (CC')^{-1}CQC'(CC')^{-1} \quad (3.3.15)$$

In the following, discrete suboptimal control laws based on modal aggregation will be reviewed.

B. Modal Aggregation

Discrete suboptimal control laws for modal aggregation will be reviewed for two modal aggregation methods: Davison's and Chidambara's methods.

In Davison's method, a discrete suboptimal control law can be obtained by repeating the analysis of the discrete exact aggregation but using an aggregation matrix C as given in (3.2.22).

In Chidambara's method, a discrete suboptimal control law can be obtained by following a parallel analysis to that of the continuous time case reviewed earlier.

Consider the simplified system given in (3.3.8) where F, and G are given in (3.2.23) and (3.2.24) but using $V=(I_{n-r}-\Lambda_2)^{-1}\Gamma_2$. It is desired to minimize the cost functional J_m associated with the reduced order model (3.3.8). J_m can be found by substituting the partitioned form of Q and the discrete version of (3.2.27) into (3.3.2). J_m is then given as

$$J_m = \sum_{0}^{\infty} (x_1'Q_1x_1 + 2x_1'Su + u'R_1u) \qquad (3.3.16)$$

where Q_1, R_1 and S are as given in (3.2.29). Following the procedure of the continuous time case, (3.3.16) can be written as

133

$$J_m = \sum_{0}^{\infty} (x_1' Q_m x_1 + \bar{u}' R_1 \bar{u}) \tag{3.3.17}$$

where $\bar{u}$ and Q_m are as defined in (3.2.30) and (3.2.33) respectively. If Q_m is positive semi-definite and R_1 is positive definite then the optimal control law for (3.3.17) is given by

$$\bar{u} = -(R_1 + G'P_aG)^{-1}G'P_aFx_1 \tag{3.3.18}$$

where P_a is the solution of the discrete Ricatti equation

$$P_a^{-1} - F_o'P_aF_o + F_o'P_aG(R + G'P_aG)GP_a'F_o - Q_m = 0 \tag{3.3.19a}$$

where
$$F_o = F - GR_1^{-1}S' \tag{3.3.19b}$$

From (3.2.30) and (3.3.18), the optimal controller for the simplified model (3.3.8) is

$$u = -K_ax_1 \tag{3.3.20a}$$

where
$$K_a = -((R_1 + G'P_aG)^{-1}G'P_aF + R_1^{-1}S') \tag{3.3.20b}$$

Since an aggregation matrix for Chadimbara's model does not always exist [10], then the control law in (3.3.20) can not serve as a suboptimal control for the full order system Sd.

3.3.3 Discrete Time Suboptimal Control Via Singular Perturbation Method

Consider the discrete time singularly perturbed system given by

$$x_1(k+1) = A_{11}x_1(k) + A_{12}x_2(k) + B_1u(k); \quad x_1(0) = x_{10} \qquad (3.3.21a)$$

$$\mu x_2(k+1) = A_{21}x_1(k) + A_{22}x_2(k) + B_2u(k); \quad x_2(0) = x_{20} \qquad (3.3.21b)$$

$$y(k) = C_1x_1(k) + C_2x_2(k) \qquad (3.3.21c)$$

where $y \in R^p$ is the output vector, $x_1(k)$ and $x_2(k)$ are respectively r and $(n-r)$ dimensional state subvectors. The slow subsystem is given as [66]

$$x_s(k+1) = A_o x_s(k) + B_o u_s(k) \; ; \quad x_s(0) = x_{10} \qquad (3.3.22)$$

$$y_s(k) = C_o x_s(k) + D_o u_s(k) \qquad (3.3.23)$$

where (A_o, B_o, C_o, D_o) are as given in (3.2.38c), $x_s \in R^r$, $y_s \in R^p$, and $u_s \in R^m$ are respectively the slow subsystem state, output and input vectors. It is desired to minimize the cost functional J_s associated with the slow subsystem.

$$J_s = \sum_0^{\infty}(y_s' y_s + u_s' R u_s) \qquad (3.3.24)$$

(3.3.24) can be written in terms of x_s and u_s as

$$J_s = \sum_0^{\infty}(x_s' C_o' C_o x_s + 2u_s' D_o' C_o x_s + u_s' R_o u_s) \qquad (3.3.25)$$

where R_o is given by

$$R_o = R + D'_o D_o \qquad (3.3.26)$$

Note that (3.3.25) is similar to (3.3.16); therefore, by following a similar analysis to the analysis which led to the optimal control law given in (3.3.20), it is possible to find that the optimal control law for (3.3.22) that minimizes (3.3.24) is given as

$$u_s^* = -K_{sp} x_s \qquad (3.3.27a)$$

where $$K_{sp} = -\left((R_o + B_o' K_s B_o)^{-1} B_o' K_s A_o + R_o^{-1} D_o' C_o\right) \qquad (3.3.27b)$$

where K_s is the positive semidefinite solution to the discrete Ricatti equation

$$K_s^{-1} - A_{oo}' K_s A_{oo} + A_{oo}' K_s B_o (R_o + B_o' K_s B_o) B_o K_s' A_{oo}$$
$$- C_o'(I_p - D_o R_o^{-1} D_o') C_o = 0 \qquad (3.3.28a)$$

where $$A_{oo} = (A_o - B_o R_o^{-1} D_o' C_o) \qquad (3.3.28b)$$

133

and I_p is the identity matrix of order p.

To find the suboptimal control law for the full order system (3.3.21) then

$$u = -K_{ss}x \tag{3.3.29a}$$

where
$$K_{ss} = [K_{sp} | 0] \tag{3.3.29b}$$

The suboptimal control law depends entirely on the slow part of the system. The fast subsystem is completely ignored.

In the following section, we are going to look at another application of the reduced order models in the control design of dynamical systems, namely eigenvalue placement using reduced order models.

3.4 Eigenvalue Assignment Via Reduced Order Models

In this section, a procedure is reviewed for determining the state feedback gain matrix of a large scale dynamic system by which only r of the n system eigenvalues are relocated to predetermined locations. In order to simplify the computation involved in calculating the state feedback gain matrix required for the pole assignment, there are two available procedures:

1. Partial pole placement [70-71]. In this method a gain matrix is first evaluated for assigning the r eigenvalues of interest.

Then another gain matrix is derived from the first one which then is applied to the full order system to achieve the pole placement.

2. Pole placement based on the reduced order model [55-56][72]. In this method a reduced order model whose eigenvalues are those r eigenvalues which are to be reassigned is first derived. Then, the gain matrix required to relocate the r eigenvalues of the reduced order model to the desired positions is calculated. Then another gain matrix is derived from the first one which then is applied to the full order system to achieve the pole placement.

In this thesis, since the first approach is not related to our work, it will not be considered any more.

In this section, pole placement via reduced order models based on aggregation and singular perturbation will be reviewed

3.4.1 Pole Placement Via Aggregation

Here, we are going to look at a procedure developed by Rao and Lamba [55] by which it is possible to reassign an r eigenvalues of the full order system based on the reduced order one.

Consider the full order system $S1$ of order n given in (3.2.1). It is desired to find a state feedback gain matrix which will relocate an r eigenvalues of the system of (3.2.1) to specified new locations while the remaining $(n-r)$ eigenvalues should not be disturbed. To achieve this purpose, a reduced order model of order r is first derived from the full order system using an appropriate aggregation matrix C. Define the aggregation matrix C as [55].

$$C = T_1 M^{-1} \tag{3.4.1}$$

where $T_1 = [I_r | 0]$, I_r is the identity matrix of order r and M is the modal matrix of A. M is arranged in away such that the first r columns correspond to the r eigenvalues which are to be relocated. Note that these r eigenvalues do not have to be the dominant ones.

Based on the reduced order model $S2$ given in (3.2.8), the gain matrix K_p is calculated via a pole placement algorithm such that the r eigenvalues of the reduced order system are shifted to the desired locations. In this thesis, Kautsky and Nichols algorithm [73] will be used for the pole placement. The control law that achieves the placement is then given by

$$u = -K_p x_1 \tag{3.4.2}$$

Applying the control law in (3.4.2) to (3.2.8) the reduced order closed loop system is obtained as

$$\dot{x}_1 = (F - GK_p)x_1 \tag{3.4.3}$$

Substituting (3.2.13) into (3.4.2) a control law for the full order system can be obtained as

$$u = -K_pCx \tag{3.4.4}$$

Applying the control law in (3.4.4) to (3.2.1) the closed loop system is obtained as

$$\dot{x} = (A - BK_pC)x \tag{3.4.5}$$

The following theorem gives the main result [55]

<u>Theorem 3.4</u>

If the aggregation matrix C is selected as in (3.4.1), then the eigenvalues of the matrix $A-BK_pC$ are the union of the eigenvalues of the matrix $F-GK_p$ and the $(n-r)$ undisturbed eigenvalues of the matrix A.

From the theorem we see that then it is possible to achieve the design purpose.

133

3.3.2 Pole Placement Via Singular Perturbation

Consider the singularly perturbed system given in (3.2.37), the slow subsystem given in (3.2.38), and the fast subsystem given by [63]

$$\mu \dot{x}_f = A_{22} x_f + B_2 u_f \tag{3.4.6}$$

where $x_f \in R^{n-r}$ and $u_f \in R^m$ are respectively the fast subsystem state and input vectors. If it is desired to reassign the slow eigenvalues or the fast eigenvalues of the full order system to new positions, then a control law is first obtained from the corresponding subsystem and then applied to the full order system.

To reassign the slow subsystem eigenvalues, a feedback matrix K_{ps} via Kautsky and Nichols algorithm is first calculated such that the eigenvalues of $(A_o - B_o K_{ps})$ are at the desired locations. For $\mu \ll 1$, the control law u that should be applied to the full order system (3.2.37) to approximately achieve the placement is given by [72]

$$u = -K_{ss} x \tag{3.4.7a}$$

where $$K_{ss} = [K_{ps} | 0] \tag{3.4.7b}$$

The resulting closed system is

$$\dot{x} = (A - BK_{ss})x \tag{3.4.8}$$

The resulting closed loop eigenvalues will be $1/\mu$ times the first order perturbations of the desired ones, i.e., if λ_s is a desired eigenvalue, then the corresponding closed loop eigenvalue will be $[\lambda_s + O(\mu)]/\mu$ [72].

To reassign the fast eigenvalues, then a feedback gain matrix K_{pf} is first calculated such as the eigenvalues of $(A_{22} - B_2 K_{pf})$ are at the desired locations. Then a control law u is applied to (3.2.37) to obtain the closed loop system

$$\dot{x} = (A - BK_{fs})x \tag{3.4.9a}$$

where

$$K_{fs} = [K_f | 0] \tag{3.4.9b}$$

Again, the resulting closed loop eigenvalues will be $1/\mu$ times the first order perturbations of the desired ones.

Pole placement in discrete time systems is done exactly the same way as in the continuous time systems and same results are obtained. Therefore, pole placement for discrete time systems will not be repeated.

3.5 Summary

In this chapter, the optimal control law was reviewed. It was seen that if the system dimension is large, then the computation and the implementation of the optimal control law may be impractical. Therefore, suboptimal control laws based on reduced order systems have become important to study. Reduced order models obtained by aggregation and singular perturbation reduction methods were used to find the suboptimal control laws.

Another area in which reduced order models are applied is pole placement. If it is desired to relocate only r eigenvalues of the system then by properly selecting the aggregation matrix, it is possible to achieve the placement accurately. However, in singular perturbation, it is only possible to relocate the slow or the fast subsystem eigenvalues. Moreover, the resultant closed loop system eigenvalues will only approximate the desired eigenvalues.

In the next chapter, we are going to present a new suboptimal LQR control law based on balancing. Also, a suboptiml control law based on a new reduction method taking both features of balancing and aggregation will be derived and evaluated. Eigenvalue placement using balanced reduced order systems and the new reduction method will be given in chapter 5.

Chapter 4

Suboptimal Control Based on Balancing: Simulation and Results

4.1 Introduction

Balancing reduction method has been considered a very valuable tool to achieve model reduction. Since the reduction is based on the controllability and the observability measures of the system, the reduced order model approximates, to a great deal, the full order system. During the last decade, there has been an extensive literature on balancing. Most of the literature has dealt with the open loop characteristics of the balanced system and the reduced order model.

The use of balancing in the design of reduced order controllers has also been considered. It has been applied to the reduction of the full order controller to a smaller one. However, no work has been done towards the design of reduced order controllers based on reduced order models from the point of view of suboptimal control.

133

In this chapter, we are going to introduce a new suboptimal control law based on reduced order models obtained by balancing. Also, we are going to compare the characteristics of the suboptimal closed loop system obtained by balancing to those obtained by aggregation and singular perturbation.

The chapter is divided as follows: in section 4.2, the new suboptimal control law based on balancing will be introduced. The performance of the suboptimal closed loop systems based on balancing, aggregation, and singular perturbation reduction methods will be analyzed in section 4.3. A new reduction method which utilizes features of balancing and aggregation, called Balagg, will be introduced in section 4.4. Sections 4.5 and 4.6 will compare suboptimal control laws based on balancing, balagg and aggregation through simulation of different practical examples for continuous and discrete time cases respectively. Also, in section 4.7 and 4.8, the comparison will be done between balancing and singular perturbation for continuous and discrete time cases respectively. Section 4.9 will give the chapter summary.

4.2 Suboptimal Control Law for Continuous Time Balanced Systems

Consider the controllable, observable, and stable system SI given as

$$SI: \quad \dot{x}(t) = Ax(t) + Bu(t) \qquad (4.2.1a)$$

$$y(t) = Hx(t) \tag{4.2.1b}$$

A balanced representation of Sl is given as

$$Sb: \quad \dot{x}_b(t) = A_b x_b(t) + B_b u(t) \tag{4.2.2a}$$

$$y(t) = C_b x_b(t) \tag{4.2.2b}$$

where

$$A_b = TAT^{-1}; \; B_b = TB; \; C_b = HT^{-1}; \; x_b(t) = Tx(t) \tag{4.2.3}$$

The controllability and observability grammians of Sb are equal to Σ. The balanced system Sb and the grammian Σ can be partitioned as

$$\begin{bmatrix} \dot{x}_{b1} \\ \dot{x}_{b2} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_{b1} \\ x_{b2} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \tag{4.2.4a}$$

$$y = [C_1 \quad C_2] \begin{bmatrix} x_{b1} \\ x_{b2} \end{bmatrix} \tag{4.2.4b}$$

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \tag{4.2.4c}$$

where $\Sigma_1 = diag(\sigma_1, \ldots, \sigma_r)$ and $\Sigma_2 = diag(\sigma_{r+1}, \ldots, \sigma_n)$. If $\sigma_r/\sigma_{r+1} \gg 1$, then the subsystem

$$\dot{x}_1 = A_{11} x_1 + B_1 u \tag{4.2.5a}$$

$$\hat{y} = C_1 x_1 \tag{4.2.5b}$$

is the reduced order model of the full order balanced system which will contain only the most controllable and observable part of the system.

Associate with the full order unbalanced system $S1$ a cost functional J given as

$$J = \int_0^\infty (x'Qx + u'Ru)dt \qquad (4.2.6)$$

$Q \in R^{nxn}$ is positive semi-definite and $R \in R^{mxm}$ is positive definite. It is desired to obtain an optimal control law that minimizes the cost functional J_{bm} associated with the reduced order model (4.2.5) and which can be used as a suboptimal control law to the full order balanced system Sb. To find the suboptimal control law, it is necessary first to express the cost functional J given in (4.2.6) in terms of the balanced coordinate. To do this, we substitute (4.2.3) into (4.2.6). The cost functional in balanced coordinate is then given as

$$J = \int_0^\infty \left((T^{-1}x_b)'Q(T^{-1}x_b) + u'Ru\right)dt \qquad (4.2.7a)$$

$$J = \int_0^\infty (x_b'(T^{-1})'QT^{-1}x_b + u'Ru)dt \qquad (4.2.7b)$$

Define $Q_b = (T^{-1})'QT^{-1}$ to be the weighting matrix in balanced coordinate system. Then

$$J = \int_0^\infty (x_b' Q_b x_b + u'Ru)dt \tag{4.2.8}$$

is the cost functional in balanced coordinate. Let the cost functional that gives the optimal control law for the balanced reduced order system given in (4.2.5) be

$$J_{bm} = \int_0^\infty (x_1' Q_1 x_1 + u'Ru)dt \tag{4.2.9}$$

To find Q_1, we should note that the contribution of a state to the cost functional depends on how strongly controllable as well as how strongly observable it is [60]. Therefore, a natural way to obtain Q_1 is to delete from Q_b all the weights associated with the weakly controllable and observable states, i.e., we keep only the weights associated with the r states that are kept in the reduced order model. Let

$$Q_b = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \tag{4.2.10}$$

then

$$Q_1 = Q_{11} \tag{4.2.11}$$

The optimal control law for (4.2.5) is given as

$$u = -K_b x_1 \tag{4.2.12}$$

where K_b is

$$K_b = R^{-1}B_1'P_b \tag{4.2.13}$$

and P_b is the solution of the reduced order Riccati equation

$$A_{11}'P_b + P_b'A_{11} - P_bB_1R^{-1}B_1'P_b + Q_1 = 0 \tag{4.2.14}$$

Now, it is desired to use the optimal control law (4.2.12) of the reduced order model as a suboptimal control law for the full order balanced system (4.2.2). The suboptimal control is found as follows

$$u = -K_b x_1 \cong K_b[I_r|0]x_b = [K_b|0]x_b \tag{4.2.15a}$$

$$u \cong -K_{sb}x_b \tag{4.2.15b}$$

where $K_{sb} = [K_b|0]$ is the suboptimal gain. The approximation of x_1 by $[I_r|0]x_b$ is justified because x_1 is the state solution of (4.2.5) which represents the most controllable and observable subsystem and x_{b1} contains the most controllable and observable states of the full order system.

Applying the control law in (4.2.15) to (4.2.2), the following suboptimal closed loop system is obtained

$$\dot{x} = (A_b - B_bK_{sb})x \tag{4.2.16}$$

133

4.3 Performance Evaluation

In general, for any suboptimal control law to be considered as a good control law for the full order system, it is important that the performance characteristics of the suboptimal closed loop system be very close to those of the optimal closed loop system. It is very important to preserve closed loop stability, to have acceptable degradation in the suboptimal cost functional value with respect to the optimal one, and to make the differences between the optimal and the suboptimal closed loop responses and controls acceptable. Also, a desirable property is to have the eigenvalues of the suboptimal closed loop system resulting form applying the suboptimal control law be very close to those of the optimal closed loop system obtained from applying the optimal control law. In the following these different characteristics will be discussed for suboptimal closed loop systems based on aggregation, singular perturbation and balancing

1. Stability

<u>Theorem 4.1 [67]</u>

If a linear time invariant system S is stabilizable and detectable, then the optimal control law obtained as in (3.2.3) will give a stable closed loop system, i.e $A-BK$ is stable.

In exact and modal aggregation, since $S1$ is assumed to be stable, controllable, and observable and since the aggregated matrix F inherits r eigenvalues of $S1$ then it follows that the reduced order system $S2$ is always stable. Therefore, it follows from theorem (4.1) that the optimal control law derived for the reduced order model $S2$ gives a stable closed loop system, i.e $F-GK_a$, where K_a in general is as defined in (3.2.11),is stable. In a singularly perturbed system, it follows from theorems (3.1) and (4.1) that the control law (3.2.42) is stabilizing for the slow subsystem (3.2.38). In balancing, since by assumption, Sb is stable, controllable, and observable, then the reduced order system (4.2.5) is stable, controllable, and observable. This is because in balancing, as mentioned earlier, every subsystem is stable, controllable and observable. Therefore, $A_{11}-B_1K_b$ is stable where K_b is as given in (4.2.13).

In suboptimal control, however, we are interested in the stability of the suboptimal closed loop system, i.e. for aggregation we would like to check the stability of $A-BK_{sa}$ where K_{sa} in general is as defined in (3.2.20b) and for singularly perturbed systems the stability of $A-BK_{ss}$, where K_{ss} is as defined in (3.2.44b), is of interest. Also, in balancing the stability of $A_b-B_bK_{sb}$ should be checked where K_{sb} is as given in (4.2.15).

The stability of the suboptimal closed loop system based on aggregation is always guaranteed if the full order system is stable [55]. In [72], it is claimed that the stability of the suboptimal closed loop system based on singular perturbation is guaranteed if (A_{22}) is stable where (A_{22}) is the fast subsystem matrix. Also, Chow and Kokotovic' in [63] implied by theorem 4 of [63] that the suboptimal system is stable if (A_{22}) is stable. However, it will be seen in this thesis that the suboptimal system is not always stable. The reason is that the stability result of [63] is only asymptotic ($\mu \to 0$) and hence is valid for a range of μ as discussed in [57]. In balancing, extensive simulation studies showed that suboptimal control law given in (4.2.15) results almost generically in a stable suboptimal closed loop system.

2. Performance Degradation

When a suboptimal control law is applied to the open loop system, the value of the cost functional will be larger than that of the optimal one and never smaller. In fact, the suboptimal cost value can be infinite if the suboptimal control law is not stabilizing. A measure of how close the suboptimal cost value to the optimal one is given by the suboptimality degree.

Definition 4.1 [60]

A suboptimal control law is said to be suboptimal with degree ϵ for the open loop system with respect to the optimal one if there exists a positive number ϵ such that

$$\tilde{J}(x_0) \le \epsilon^{-1} J^*(x_0) \qquad \forall x_0 \qquad (4.3.1)$$

where J^* is the optimal cost and $\tilde{J}$ is the suboptimal cost which will be defined later in this section.

Theorem 4.2 [60]

If a non-optimal control law stabilizes the open loop system, then it is a suboptimal control law for the open loop system with degree

$$\epsilon = \lambda_{max}^{-1}(LP^{-1}) \tag{4.3.2}$$

where P is the solution of the Riccati equation given in (3.2.5), L is the unique symmetric positive definite solution of the Lyapunov equation

$$Aggregation: \quad (A - BK_{sa})'L + L(A - BK_{sa}) + Q + K_{sa}'RK_{sa} = 0 \tag{4.3.3a}$$

$$Singular\,Perturbation: \quad (A - BK_{ss})'L + L(A - BK_{ss}) + C'C + K_{ss}'RK_{ss} = 0 \tag{4.3.3b}$$

$$Balancing: \quad (A_b - B_bK_{sb})'L + L(A_b - B_bK_{sb}) + Q_b + K_{sb}'RK_{sb} = 0 \tag{4.3.3c}$$

and λ_{max} is the largest eigenvalue of the indicated matrix. Also, a similar result has been obtained in [69].

Another measure of performance degradation is given by Sinha and Bruin [74] as the difference in percentage between the suboptimal cost functional value $\bar{J}$ and the optimal one J^*.

$$dJ = 100*(\bar{J} - J^*) / J^* \tag{4.3.4}$$

For aggregation $\bar{J}$ is given by [67]

$$\bar{J} = x_0' L x_0 \tag{4.3.5}$$

where L is the solution to (4.3.3a) and x_0 is the initial condition vector. For singular perturbation, $\bar{J}$ is given by [63]

$$\bar{J} = x_0' L x_0 \tag{4.3.6}$$

where L is the solution to (4.3.3b) and for balancing, it is given by

$$\bar{J} = x_{b0}' L x_{b0} \tag{4.3.7}$$

where L is the solution to (4.3.3c) and x_{b0} is the initial condition vector in balanced form.

133

3. Suboptimal Response and Eigenvalue Position

As was seen in section (3.4.1), for a class of aggregation matrices C, the suboptimal eigenvalues are the union of the optimal eigenvalues of the reduced order system and the remaining eigenvalues of A which are not eigenvalues of F. However, the suboptimal eigenvalues could be completely or partially different from the optimal ones. There is no formula or equation that relates the suboptimal to the optimal eigenvalues. Therefore, the optimal response can be different from the suboptimal one.

Also, in singular perturbation there is no relation between the suboptimal and the optimal eigenvalues. Therefore, the optimal response can be different from the suboptimal one.

In balancing, there is also no equation that relates the optimal to the suboptimal eigenvalues. However, since the suboptimal control law is based on the most controllable and observable states of the system, one would expect the optimal and the suboptimal responses to be close to each other.

4.4 Suboptimal Control Based on Balancing and Aggregation

Suboptimal control based on balancing as derived in section 4.2 is expected to perform very well due to the fact that the design is completely based on the most controllable and observable part of the system. However, although the extensive simulation which have been done on numerous practical systems showed that the resulting suboptimal closed loop systems had always been stable, no theory that guarantees the stability of the suboptimal closed loop system had been obtained. Therefore, to overcome this problem, we propose here a new suboptimal method which will take an advantage of balancing and guarantees stability of the resulting suboptimal closed loop system.

The proposed method is a combination of balancing and aggregation. Given an unbalanced system Sl, it is first transformed into a balanced system Sb, then a reduced order system is obtained from Sb using aggregation technique based on Davison. The reduced order system will have the dominant eigenvalues of Sb as its eigenvalues. A suboptimal control law based on the aggregated system is then obtained as outlined in chapter 3. The resulting suboptimal closed loop system will take an advantage of balancing and in the same time it will be stable. In the following, we are going to refer to this method as Balagg.

4.4.1 Reduced Order Model

Consider the unbalanced system Sl given by (4.2.1) and the balanced system Sb given by (4.2.2). The balanced system is related to the unbalanced system by the similarity transformation as defined in (4.2.3). It is known that similarity transformation does not change the eigenvalues i.e.

133

the eigenvalues of Sl and Sb are the same. However, it can be shown that the modal matrices of the systems are affected and they are related by T as

$$M_b = TM \tag{4.4.1}$$

where M_b is the modal matrix of the balanced system and M is the modal matrix of the unbalanced one. M_b, M, and T can be partitioned as

$$M_b = \begin{bmatrix} M_{b11} & M_{b12} \\ M_{b21} & M_{b22} \end{bmatrix} \tag{4.4.2a}$$

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{4.4.2b}$$

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \tag{4.4.2c}$$

From (4.4.1) and (4.4.2), it follows that M_{b11} is given by

$$M_{b11} = T_{11}M_{11} + T_{12}M_{21} \tag{4.4.3}$$

Let the reduced order model $S2$ be as obtained by Davison and presented earlier in chapter 2 as

$$S2: \quad \dot{x}_1(t) = Fx_1(t) + Gu(t) \qquad (4.4.4a)$$

$$\hat{y}(t) = Dx_1(t) \qquad (4.4.4b)$$

where $$F = M_o T_1 \Lambda T_1' M_o^{-1} \; ; \; G = M_o T_1 M^{-1} B \qquad (4.4.5a,b)$$

where $T_1 = [I_r | 0]$ and M_o is M_{11} if the modal matrix M is arranged according to the states which are to be retained in $S2$. Also, the aggregation matrix C is given by

$$C = M_{11} T_1 M^{-1} \qquad (4.4.6)$$

Now, if the aggregated system $S2$ is obtained from the balanced system Sb, then M_{b11} and M_b should be used in place of M_{11} and M respectively in (4.4.5). Substituting the value of M_{b11} (i.e M_o) as given in (4.4.3) in the aggregated system obtained from the balanced system, then F and G can be written as

$$F = (T_{11}M_{11} + T_{12}M_{21}) \; T_1 \Lambda T_1' \; (T_{11}M_{11} + T_{12}M_{21})^{-1} \qquad (4.4.7a)$$

$$G = (T_{11}M_{11} + T_{12}M_{21}) \; T_1 \; (TM)^{-1} B_b \qquad (4.4.7b)$$

Substituting B_b from (4.2.3) and simplifying, G can be written as

$$G = (T_{11}M_{11} + T_{12}M_{21})\ T_1\ M^{-1}B \tag{4.4.7c}$$

also the aggregation matrix is

$$C = (T_{11}M_{11} + T_{12}M_{21})\ T_1\ M^{-1}T^{-1} \tag{4.4.7d}$$

one advantage of this reduction method is that when M_{11}^{-1} (i.e. M_o^{-1} in (4.4.5)) does not exist, it is possible that M_{b11}^{-1} exists. Another advantage is that the reduced order system is derived from a balanced system. After deriving the reduced order model, a suboptimal control law based on Davison's method can then be derived. The resulting suboptimal closed loop system has the advantage of being stable.

4.5 Comparison of Continuous Suboptimal Control Laws Based on Balancing and Aggregation: Simulation and Results

In this section, we are going to compare suboptimal control laws obtained by balancing, aggregation and balagg. The comparison is done through simulation of practical examples. For balancing, the reduced order model given in (4.2.5) is used and the control law in (4.2.15) is used as the suboptimal control law. For aggregation, Davison's model (4.4.4) and (4.4.5) is used as the reduced order model and the control law in (3.2.20) and (3.2.22) is used as the suboptimal control law. For balagg, the system is first balanced and then the techniques of aggregation based on Davison, as given above, are used to obtain the suboptimal control law.

In every example, controllers of different orders will be obtained. The base of the comparison will be the optimal characteristics of the full order system. MatLab package has been used in the simulation.

4.5.1 Boiling Water Reactor

This is an example of a ninth order boiling water reactor (BWR) considered in [75]. The A, B, and H matrices of the system are given below.

$$A = \begin{bmatrix} -13.6 & 7.7 & -26.2 & -28.1 & 75.6 & -14.0 & -19.0 & 0.0 & 0.0 \\ 0.316 & -0.077 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 29.7 & 0.0 & -0.152 & 0.0 & 0.0 & 0.0 & 0.019 & 0.0 & 0.0 \\ 15.2 & 0.0 & 0.0 & -0.152 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.067 & -4.8 & 0.035 & -0.041 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.265 & 0.0 & 0.0 & 0.265 \\ 0.0 & 0.0 & 0.0 & 0.09 & -0.093 & 0.045 & -0.075 & 0.0045 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.163 & -1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.222 & 0.0 & 0.131 & -0.082 & -0.111 \end{bmatrix} \tag{4.5.1a}$$

$$B' = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 2.87e-5 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.13 & 1.45 & 0.0 & 0.0 \end{bmatrix} \tag{4.5.1b}$$

$$H = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 12.247 \end{bmatrix} \tag{4.5.1c}$$

Also, the initial condition vector x_0 and the weighting matrix Q are

$$x_0' = [0.5 \;\; 0.5 \;\; 0.5 \;\; 0.5 \;\; 0.5 \;\; 0.5 \;\; 0.5 \;\; 0.5 \;\; 0.5] \qquad (4.5.1d)$$

$$Q = I_n, \; where \; n = 9 \qquad (4.5.1e)$$

The open loop eigenvalues eva and the singular values of the balanced system Σ are:

eva = [-4.2427e-2 -7.6705e-2 -1.5722e-1 -2.1221e-1±3.0834e-2i -1.0006e+0 -4.7337e+0 -6.8984e+0±3.4024e1i]

Σ = [2.4600e+1 6.7655e+0 3.3662e-1 3.3431e-2 2.1746e-3 6.0435e-5 4.4561e-5 2.8658e-5 2.3992e-5]

Three cases will be considered : $R = I_m$, $R = 20 * I_m$ and $R = 0.1 * I_m$ in (4.2.6) where I_m is the identity matrix of order $m=2$. In every case the full order system will be reduced to a fifth and a second order system using balancing, aggregation, and balagg. Then, the suboptimal control laws obtained using the reduced order models will be applied to the full order system and the performance characteristics will be evaluated.

4.5.1.1 Simulation and Results

Tables (4.5.1-4.5.3), give a summary of the different characteristics of the optimal and the suboptimal closed loop systems using balancing, aggregation and balagg. Comparing the characteristics of the suboptimal closed loop systems obtained by balancing, aggregation, and balagg to those of the optimal closed loop system, we see the following:

1. All the methods give stable suboptimal closed loop systems.

2. The performance degradation dJ is very small for balancing. Also, in general, it is small for aggregation but not as small as balancing. Although, dJ in balagg is small but it is larger than that of aggregation only for $r=5$ but it, in general, becomes smaller for $r=2$. It is interesting to note that dJ decreases for balancing and balagg as the reduced order changes from 5 to 2. In balancing, this might be because of the retention of the most dominant singular values in the reduced order system. In balagg, it might be because that the reduced order system has retained the most dominant eigenvalues and dominant singular values. Also, the reason for this might be the fact that when $r=2$, the movement of the eigenvalues, for this case, becomes easier and hence less control effort is required. Less control results in less performance degradation and higher suboptimality degree.

3. The suboptimality degree ϵ is close to unity for balancing. For aggregation, it is good when $R=I_m$ and close to unity for $R=20*I_m$ but small when $R=0.1*I_m$. For balagg, it is very small for $R=I_m$ and $R=0.1*I_m$, but it is good when $R=20*I_m$. Also,

as in the case of dJ, ϵ improves for balancing and balagg as the reduced order changes from 5 to 2. The explanation used with dJ may be used with ϵ.

4. Closed loop eigenvalues: It can be seen that the suboptimal control based on balancing has resulted in eigenvalues very close to the optimal ones. In fact, they are almost identical for $R = 20 * I_m$. Therefore, suboptimal control based on balancing placed the eigenvalues almost optimally. However, in general, the same remark can not be said about suboptimal controls based on aggregation nor balagg.

5. Suboptimal step responses and controls: Figures (4.5.1-4.5.6) show the suboptimal step responses and controls obtained by the different reduction methods. As a basis of the comparison, the optimal responses and controls are also shown. Note that through out the thesis standard conventions will be used for the axis i.e the horizontal axis represents time and the vertical axis represents the output amplitude. From the figures, it can be seen that suboptimal responses based on balancing are completely identical to the optimal ones for all R and both $r = 2$ and $r = 5$. Also, noting the different scales for the controls, it can be seen that the suboptimal controls based on balancing are identical to the optimum ones. However, suboptimal responses and controls based on aggregation and balagg are, in general, far from optimum, but they are closer for balagg.

134

Suboptimal Balanced Characteristics	Suboptimal Aggregated Characteristics	Suboptimal Balagg Characteristics
Reduced order = 5		
J=1.0075e+1 dJ=1.2737e+0% ϵ=9.0367e-1	J=1.0645e+1 dJ=6.9991e+0 % ϵ=7.6971e-1	J=1.7831e+1 dJ=7.9236e+1% ϵ=1.9453e-1
Eigenvalues		
-6.8984e+0±3.4024e+1 i -4.7337e+0 -1.7651e+0 -2.0507e-1±5.3839e-2 i -1.5241e-1±1.4047e-2 i -7.6831e-2	-6.8984e+0±3.4024e+1 i -4.7337e+0 -1.0006e+0 -2.3220e-1±7.7951e-2 i -7.6884e-2 -1.6609e-1±1.5996e-2 i	-6.8984e+0±3.4024e+1 i -4.7337e+0 -1.0006e+0 -3.1211e-1 -1.8919e-±6.1889e-2 i -1.5546e-1 -7.6419e-2
Reduced order = 2		
J=9.9548e+0 dJ=6.4399e-2 % ϵ=9.6468e-1	J=1.0048e+1 dJ=9.9726e-1 % ϵ=7.2045e-1	J=1.0140e+1 dJ=1.9307e+0 % ϵ=3.2146e-1
Eigenvalues		
-6.8984e+0±3.4024e+1 i -4.7337e+0 -1.7640e+0 -2.1480e-1±3.6674e-2 i -1.4568e-1±5.4487e-3 i -7.6870e-2	-6.8984e+0±3.4024e+1 i -4.7337e+0 -1.0006e+0 -2.1221e-1±3.0838e-2 -7.6888e-2 -1.5151e-1 -1.5722e-1	-6.8984e+0±3.4024e+1 i -4.7337e+0 -1.0006e+0 -2.5335e-1 -2.1221e-1±3.0834e-2 i -1.5722e-1 -7.7561e-2

Optimal Characteristics:

J^* = 9.9484e+0

Eigenvalues = [-6.8986e+0±3.4024e+1i -4.7337e+0 -1.7653e+0 -2.0713e-1±5.5583e-2i -1.5022e-1±1.7112e-2i -7.6949e-2]

Table 4.5.1: BWR Suboptimal Characteristics Based on Balancing, Aggregation, and Balagg for $R = I_m$

0093

Suboptimal Balanced Characteristics	Suboptimal Aggregated Characteristics	Suboptimal Balagg Characteristics
Reduced order = 5		
J=1.5409e+1 dJ=2.1721e-1% ϵ=9.7482e-1	J=1.5379e+1 dJ=2.3027e-2% ϵ=9.7575e-1	J=1.6531e+1 dJ=7.5199e+0% ϵ=6.5322e-1
Eigenvalues		
-6.8984e+0±3.4024e+1 i -4.7337e+0 -1.0517e+0 -2.1104e-1±3.3461e-2 i -1.5662e-1 -7.6451e-2 -6.8391e-2	-6.8984e+0±3.4024e+1 i -4.7337e+0 -1.0006e+0 -2.1123e-1±3.3130e-2 i -6.6905e-2 -7.6390e-2 -1.5632e-1	-6.8984e+0±3.4024e+1 i -4.7337e+0 -1.0006e+0 -2.1152e-1± 2.6861e-2 i -1.5762e-1 -7.1675e-2 -7.8003e-2
Reduced order = 2		
J=1.5381e+1 dJ=3.8582e-2 % ϵ=9.9630e-1	J=1.6202e+1 dJ=5.3774e+0 % ϵ=9.2565e-1	J=1.5525e+1 dJ=9.7370e-1% ϵ=7.3448e-1
Eigenvalues		
-6.8984e+0±3.4024e+1 i -4.7337e+0 -1.0513e+0 -2.1249e-1±3.1908e-2 i -6.5766e-2 -1.5666e-1 -7.6468e-2	-6.8984e+0±3.4024e+1 i -4.7337e+0 -1.0006e+0 -2.1221e-1±3.0834e-2 i -5.3546e-2 -7.6653e-2 -1.5722e-1	-6.8984e+0±3.4024e+1 i -4.7337e+0 -1.0006e+0 -2.1221e-1±3.0834e-2 i -1.5722e-1 -7.3566e-2±2.6802e-3 i

Optimal Characteristics:

$J^* = 1.5375e+1$

Eigenvalues =[-6.8984e+0±3.4024e+1i -4.7337e+0 -1.0518e+0 -2.1123e-1±3.3933e-2i -1.5622e-1 -7.6216e-2 -6.8605e-2]

Table 4.5.2: BWR Suboptimal Characteristics Based on Balancing, Aggregation, and Balagg for $R = 20 * I_m$

Suboptimal Balanced Characteristics	**Suboptimal Aggregated Characteristics**	**Suboptimal Balagg Characteristics**
	Reduced order = 5	
J=9.1836e+0 dJ=1.7852e+0% ϵ=8.7782e-1	J=1.3183e+1 dJ=4.6113e+1% ϵ=4.7163e-1	J=3.0304e+1 dJ=2.3587e+2% ϵ=7.3126e-2
	Eigenvalues	
-6.8984e+0±3.4024e+1 i -4.7350e+0 -4.7134e+0 -2.0679e-1±6.5460e-2 i -1.6116e-1±1.4265e-2 i -7.6822e-2	-6.8984e+0±3.4024e+1 i -4.7337e+0 -1.0006e+0 -5.6282e-1 -3.0799e-1 -7.6871e-2 -1.7314e-1±5.9904e-3	-6.8984e+0±3.4024e+1 i -4.7337e+0 -1.0006e+0 -8.4971e-1 -2.0428e-1± 6.0354e-2 i -1.5582e-1 -7.6434e-2
	Reduced order = 2	
J=9.0444e+0 dJ=2.4234e-1 % ϵ=9.3683e-1	J=1.2971e+1 dJ=4.3761e+1 % ϵ=4.3233e-1	J=9.6496e+0 dJ=6.9501e+0% ϵ=1.6605e-1
	Eigenvalues	
-6.8984e+0±3.4024e+1 i -4.7348e+0 -4.7167e+0 -2.1635e-1±3.8150e-2 i -1.5570e-1±1.2731e-2 i -7.6847e-2	-6.8984e+0±3.4024e+1 i -4.7337e+0 -1.0006e+0 -4.6218e-1 -2.1221e-1±3.0834e-2 i -7.6855e-2 -1.5722e-1	-6.8984e+0±3.4024e+1 i -4.7337e+0 -1.0006e+0 -7.9173e-1 -2.1221e-1± 3.0834e-2 i -1.5722e-1 -7.7509e-2

Optimal Characteristics:

J^* = 9.0225e+0

Eigenvalues = [-6.9000e+0±3.4024e+1i -4.7242e+0±3.1255e-2i -2.0924e-1±6.6823e-2i -1.5858e-1±1.8484e-2i -7.6933e-2]

Table 4.5.3: BWR Suboptimal Characteristics Based on Balancing, Aggregation, and Balagg for $R=0.1*I_m$

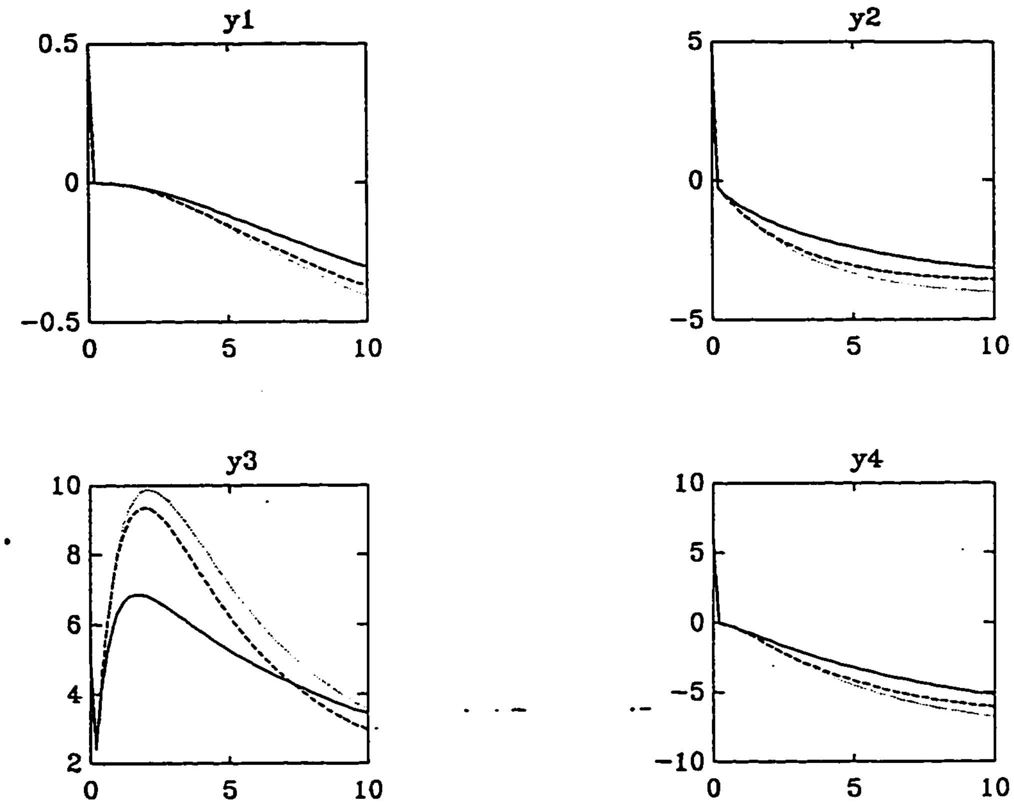

Fig. 4.5.1a Closed Loop Responses: Optimal (–), Suboptimal Balanced (–.), Aggregated (..), and Balagg (––); r=5, R=I

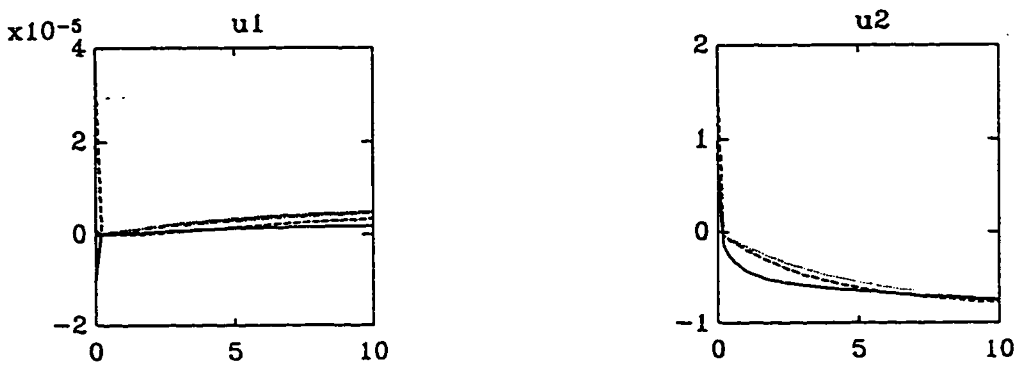

Fig.4.5.1bClosed Loop Controls: Optimal (–), Suboptimal Balanced (–.), Aggregated (..), and Balagg (––); r=5, R= I

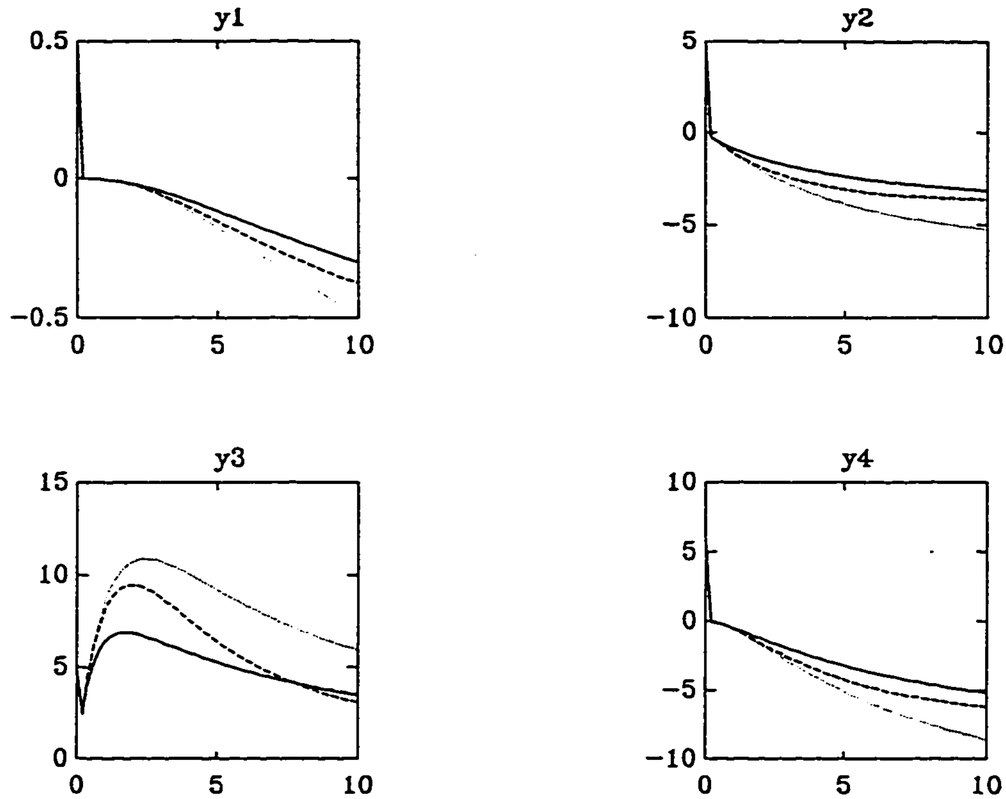

Fig. 4.5.2a Closed Loop Responses: Optimal (–), Suboptimal Balanced (–.), Aggregated (..), and Balagg (––); r=2, R=I

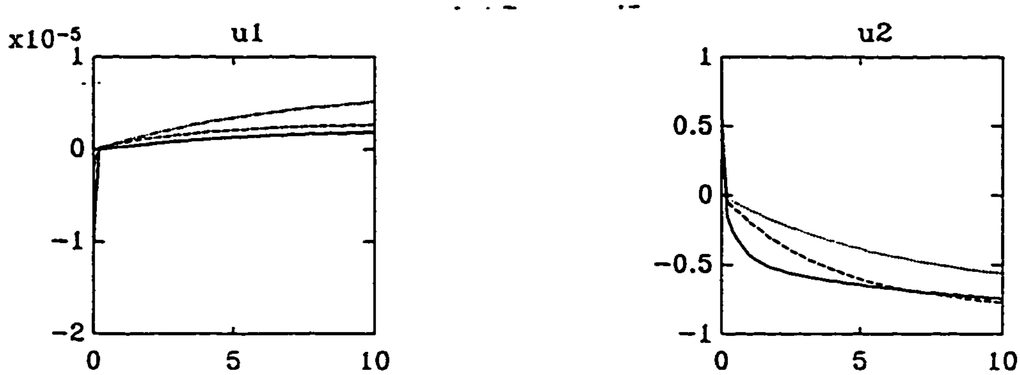

Fig4.5.2b Closed Loop Controls: Optimal (–), Suboptimal Balanced (–.), Aggregated (..), and Balagg (––); r=2, R=I

0093

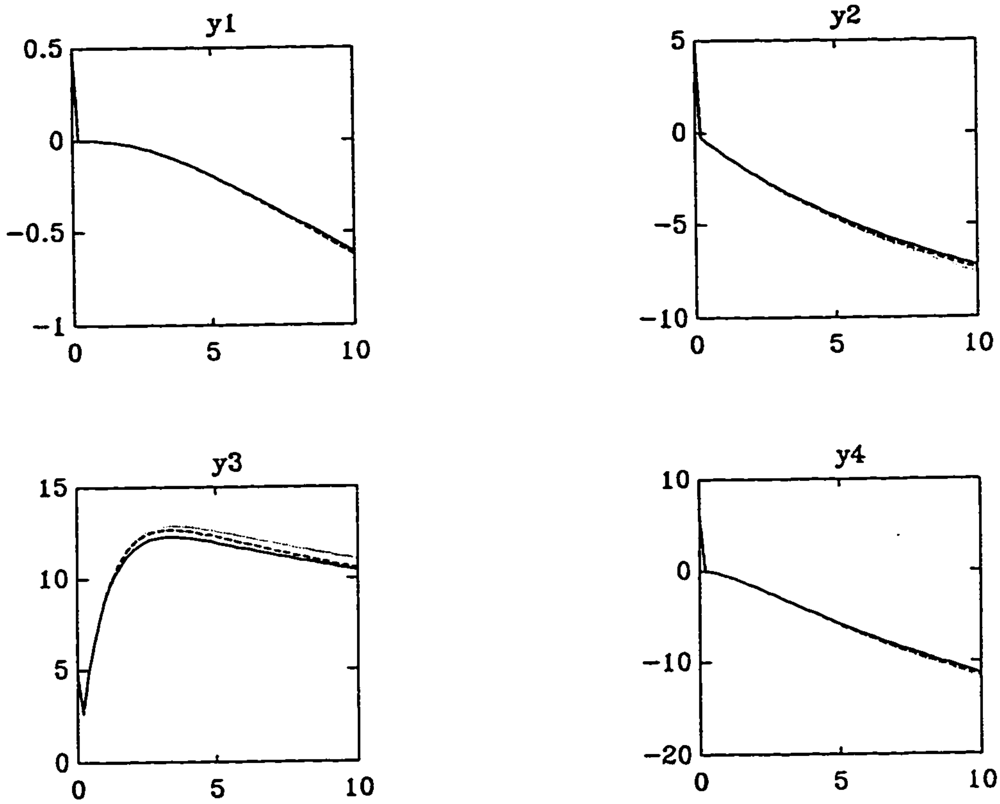

Fig.4.5.3a Closed Loop Responses: Optimal (–), Suboptimal Balanced (–.), Aggregated (..), and Balagg (––); r=5, R=20I

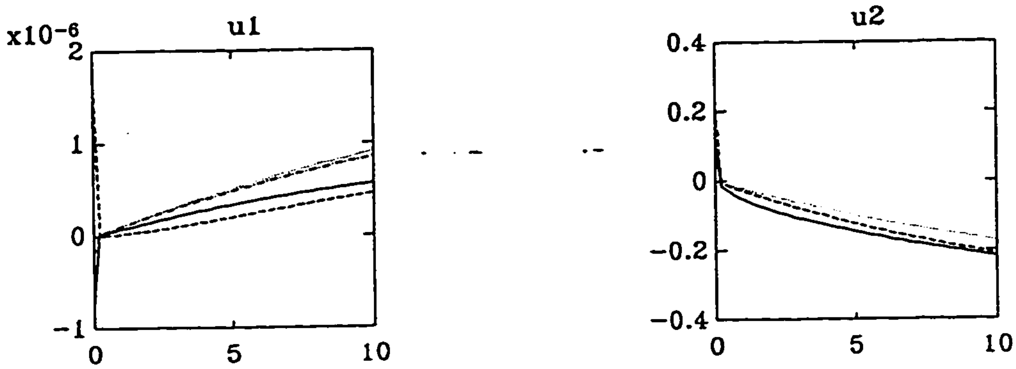

Fig.4.5.3b Closed Loop Controls: Optimal (–), Suboptimal Balanced (–.), Aggregated (..), and Balagg (––); r=5, R=20I

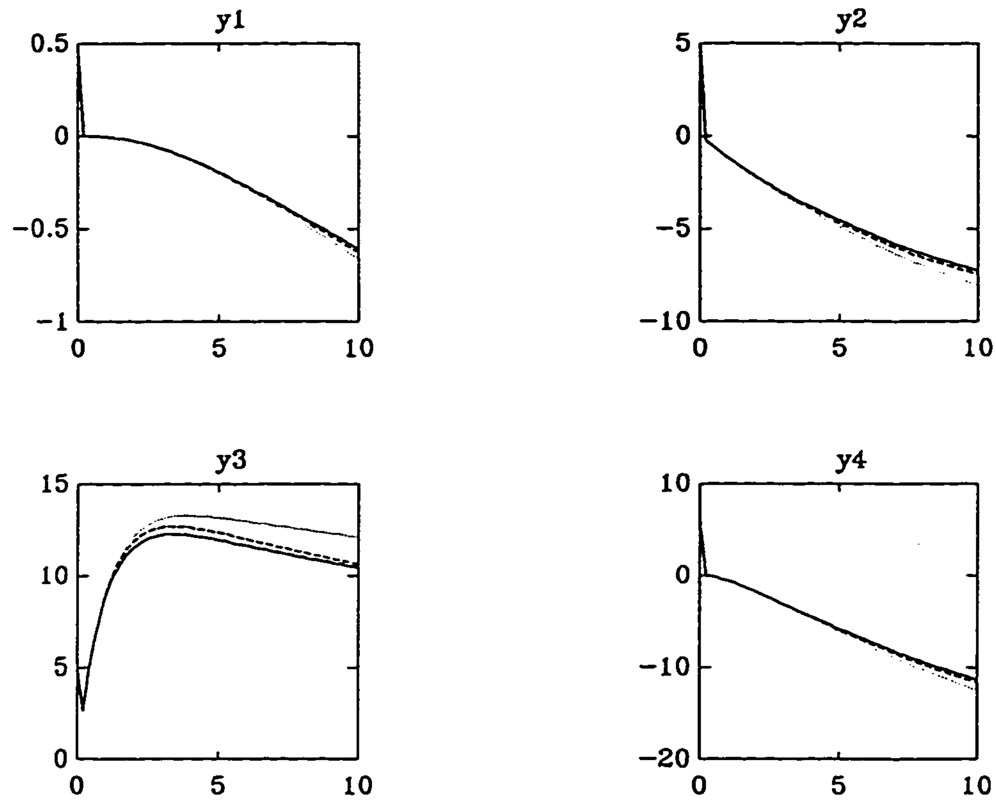

Fig.4.5.4aClosed Loop Responses: Optimal (–), Suboptimal Balanced (–.), Aggregated (..), and Balagg (––); r=2, R=20I

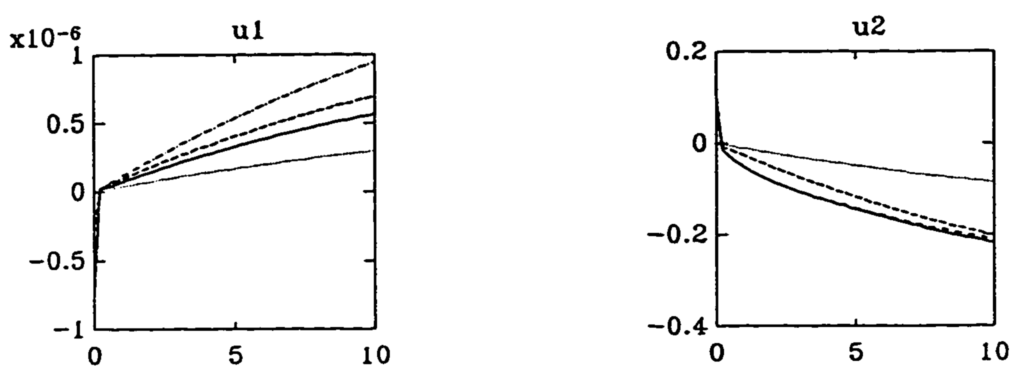

Fig.4.5.4bClosed Loop Controls: Optimal (–), Suboptimal Balanced (–.), Aggregated (..), and Balagg (––); r=2 , R= 20I

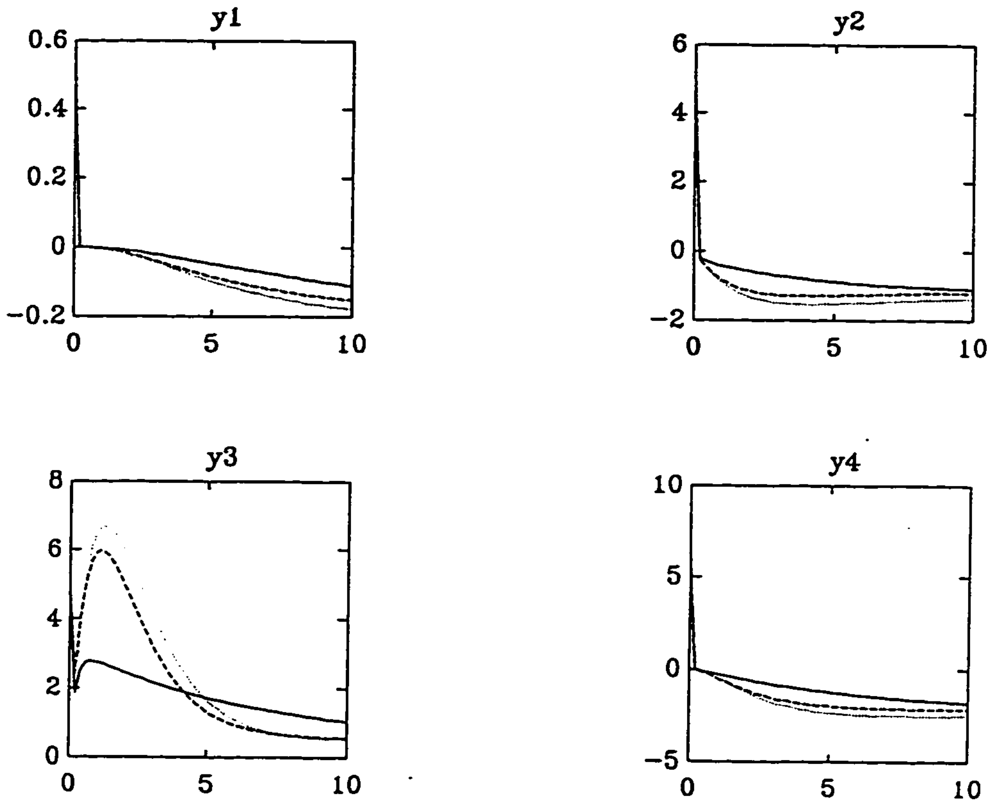

Fig.4.5.5a Closed Loop Responses: Optimal (–), Suboptimal Balanced (–.), Aggregated (..), and Balagg (––); r=5, R=0.1I

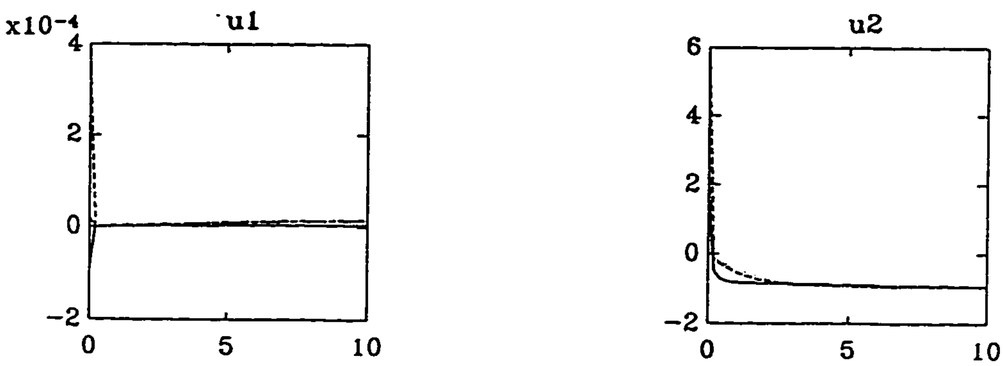

Fig.4.5.5b Closed Loop Controls: Optimal (–), Suboptimal Balanced (–.), Aggregated (..), and Balagg (––); r=5, R=0.1I

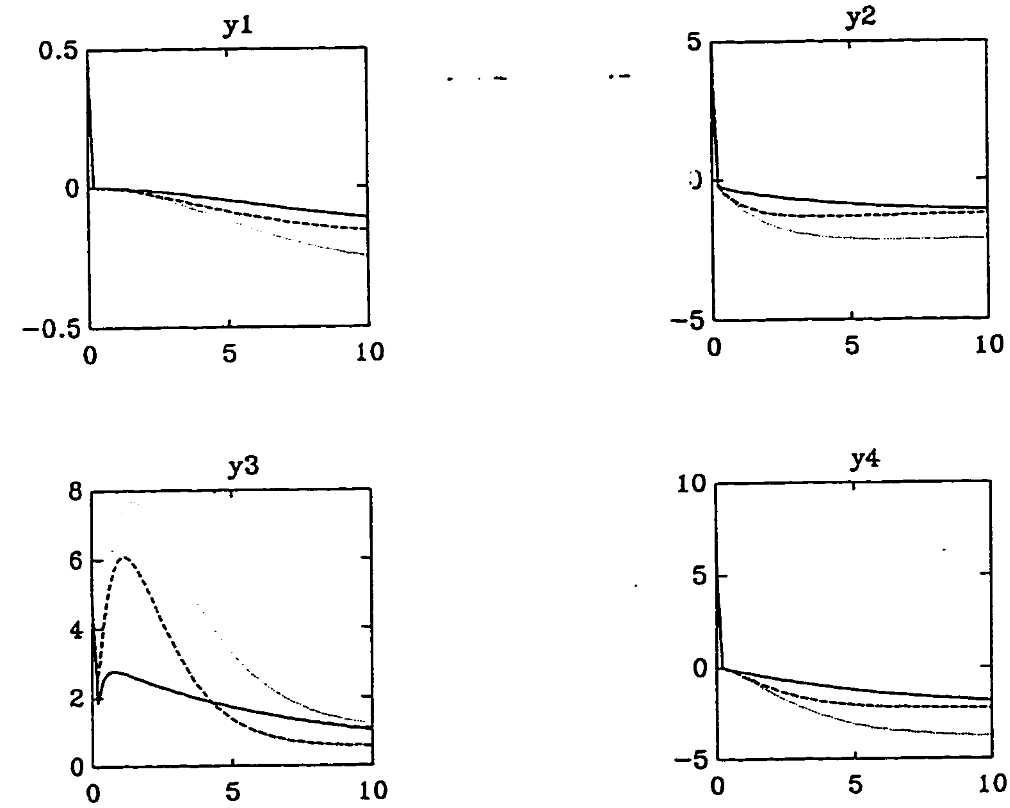

Fig.4.5.6a Closed Loop Responses: Optimal (−), Suboptimal Balanced (−.), Aggregated (..), and Balagg (−−); r=2, R= 0.1I

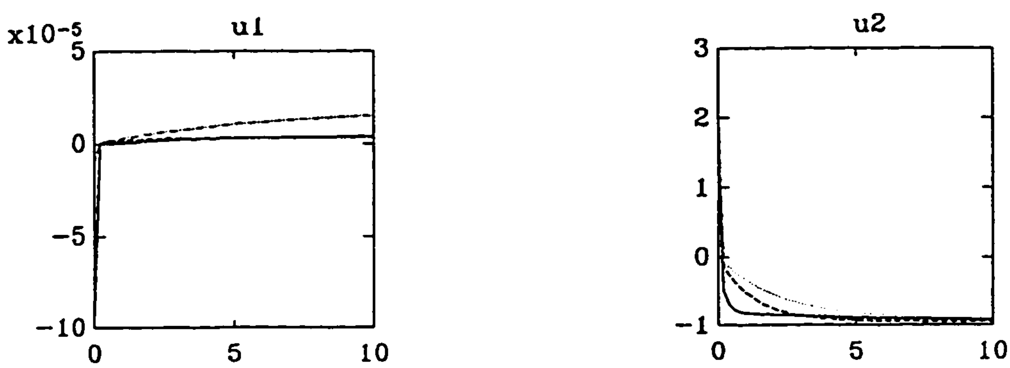

Fig. 4.5.6b Close Loop Controls: Optimal (−), Suboptimal Balanced (−.), Aggregated (..), and Balagg (−−); r=2, R= 0.1I

4.5.2 Boiler

This is an example of a ninth order boiler considered in [76]. The A, B, and H matrices of the system are given below.

$$A = \begin{bmatrix} -0.910 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4.449 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -10.262 & 571.479 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -571.479 & -10.262 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -10.987 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -15.214 & 11.622 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -11.622 & -15.214 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -89.874 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -502.665 \end{bmatrix} \tag{4.5.2a}$$

$$B' = [-4.336 \;\; -3.691 \;\; 10.141 \;\; -1.612 \;\; 16.629 \;\; -242.476 \;\; -14.261 \;\; 13.672 \;\; 82.187] \tag{4.5.2b}$$

$$H = \begin{bmatrix} -.422 & -0.736 & -0.00416 & 0.232 & -0.816 & -0.715 & 0.546 & -0.235 & -0.0806 \\ -0.00286 & 0.00136 & 0.117 & -0.393 & 0.00113 & -0.045 & 0.0685 & -0.0798 & -0.129 \end{bmatrix} \tag{4.5.2c}$$

Also, the initial condition vector x_0 and the weighting matrix Q are

$$x_0' = [0.5 \;\; 0.5 \;\; 0.5 \;\; 0.5 \;\; 0.5 \;\; 0.5 \;\; 0.5 \;\; 0.5 \;\; 0.5] \tag{4.5.2d}$$

$$Q = I_n, \text{ where } n = 9 \tag{4.5.2e}$$

The open loop eigenvalues eva and the singular values of the balanced system Σ are:

eva = [-1.0262e+1±5.7148e+2i -5.0266e+2 -8.9874e+1 -1.5214e+1±1.1622e+1i -9.1000e-1 -4.4490e+0 -1.0987e+1]

Σ = [1.1924e+1 4.6040e+0 6.3901e-1 1.1925e-1 1.1563e-1 2.8093e-2 1.2317e-2 2.1317e-3 5.3668e-4].

Three cases will be considered : $R=I_m$, $R=20*I_m$ and $R=0.1*I_m$. In every case the full order system will be reduced to a fourth and first order system using balancing, aggregation, and balagg. Then, the suboptimal control laws obtained using the reduced order models will be applied to the full order system and the performance characteristics will be evaluated.

4.5.2.1 Simulation and Results

Tables (4.5.4-4.5.6), give a summary of the different characteristics of the optimal and the suboptimal closed loop systems using balancing, aggregation and balagg. Comparing the characteristics of the suboptimal closed loop system obtained by balancing, aggregation, and balagg to those of the optimal closed loop system, we see the following:

1. Stability: All the methods give stable suboptimal closed loop systems.

2. Performance degradation dJ: It is very big for all the methods but it is small for balancing when $r=4$ for all R and acceptable for aggregation for $R=20*I_m$. Again, as in example (4.5.1), dJ decreases for balagg as the reduced order changes from 4 to 1.

3. Suboptimality degree ϵ: It is small when $r=4$ for balancing. For the remaining cases it is very small. Also, as in the case of dJ, ϵ improves for balagg as the reduced order changes from 4 to 1.

4. Closed loop eigenvalues: It can be seen that the suboptimal control based on balancing has resulted in eigenvalues very close to the optimal ones when $r=4$. However, they are not close for $r=1$. Suboptimal controls based on aggregation and balagg resulted in suboptimal eigenvalues far from optimum, but those of balagg are much closer than those of aggregation.

5. Suboptimal step responses and controls: Figures (4.5.7-4.5.12) show the suboptimal responses and controls obtained by the different reduction methods. When $r=4$, suboptimal responses and control based on balancing are identical to the optimal ones for all R. Also, those based on balagg are extremely close to optimum. However, suboptimal control based on aggregation resulted in having responses far from optimum but the controls were close to optimum. In fact, they are identical when $R=0.1*I_m$. When $r=1$, responses based on aggregation remained far from optimum, responses based on balancing shifted from their previous values and became not identical to optimum although they are still close to the optimum, and responses based on balagg have shown an improvement and became identical to the optimal responses. The improvement in the behavior of the suboptimal responses based on balagg

might be because that the reduced order system has retained the most dominant eigenvalue and a very dominant singular value.

Suboptimal Balanced Characteristics	Suboptimal Aggregated Characteristics	Suboptimal Balagg Characteristics
Reduced order = 4		
J=2.3568e-1 dJ=1.8752e+1% ϵ=4.5102e-1	J=3.1137e+0 dJ=1.4689e+3% ϵ=4.8171e-2	J=1.2277e+1 dJ=6.0861e+3% ϵ=8.9286e-3
Eigenvalues		
-1.0750e+1±5.7160e+2 i -5.0215e+2 -2.4413e+2 -9.0256e+1 -1.9080e+1 -9.3693e-1 -4.7562e+0 -1.0637e+1	-1.0987e+1 -8.9874e+1 -5.0267e+2 -1.2571e+1±5.7148e+2 i -1.5214e+1±1.1622e+1 i -6.6287e+0 -3.0160e+0	-1.2826e+1±5.7147e+2 i -5.0266e+2 -8.9874e+1 -1.5214e+1±1.1622e+1 i -2.2075e+1 -2.7903e+0 -1.0987e+1
Reduced order = 1		
J=6.2185e+0 dJ=3.0333e+3% ϵ=2.2741e-2	J=2.6907e+0 dJ=1.2557e+3% ϵ=4.6946e-2	J=8.4466e+0 dJ=4.1560e+3% ϵ=1.5006e-2
Eigenvalues		
-1.0249e+1±5.7148e+2 i -5.0254e+2 -9.1476e+1±4.1806e+0 i -1.6196e+1 -9.6202e+0±1.4525e+1 i -4.0986e+0	-1.5214e+1±1.1622e+1 i -1.0262e+1±5.7148e+2 i -5.0267e+2 -4.4490e+0 -1.0987e+1 -8.9874e+1 -4.4305e+0	-1.0262e+1±5.7148e+2 i -5.0267e+2 -8.9874e+1 -1.5214e+1±1.1622e+1 i -4.4490e+0 -1.2771e+1 -1.0987e+1

Optimal Characteristics:

$J^* = 1.9846e-1$

Eigenvalues = [-1.2226e+1±5.7147e+2i -5.1129e+2 -2.3948e+2 -8.9705e+1 -1.9131e+1 -9.7065e-1 -4.4586e+0 -1.1096e+1]

Table 4.5.4: Boiler Suboptimal Characteristics Based on Balancing, Aggregation, and Balagg for $R = I_m$

134
0093

Suboptimal Balanced Characteristics	Suboptimal Aggregated Characteristics	Suboptimal Balagg Characteristics
Reduced order = 4		
J=2.3091e-1 dJ=1.3211e+1% ϵ=5.0645e-1	J=3.1899e-1 dJ=5.6391e+1% ϵ=4.1382e-1	J=2.3887e+0 dJ=1.0711e+3% ϵ=5.9699e-2
Eigenvalues		
-1.0384e+1±5.7155e+2 i -5.0264e+2 -8.9692e+1 -5.2478e+1 -2.1094e+1 -9.3295e-1 -4.7188e+0 -1.0690e+1	-1.0987e+1 -8.9874e+1 -5.0267e+2 -1.0390e+1±5.7148e+2 i -1.5214e+1±1.1622e+1 i -1.3168e+0 -4.5287e+0	-1.0384e+1±5.7155e+2 i -5.0264e+2 -8.9692e+1 -5.2478e+1 -2.1094e+1 -9.3295e-1 -4.7188e+0 -1.0690e+1
Reduced order = 1		
J=3.3230e+0 dJ=1.5292e+3% ϵ=4.2270e-2	J=3.0683e-1 dJ=5.0430e+1% ϵ=4.1404e-1	J=1.6703e+0 dJ=7.1890e+2% ϵ=7.7303e-2
Eigenvalues		
-1.0259e+1±5.7148e+2 i -5.0265e+2 -8.9819e+1 -2.1452e+1±1.0640e+1 i -5.4166e+0 -1.0748e+1 -3.6516e+0	-1.5214e+0±1.1622e+1 i -1.0262e+1±5.7148e+2 i -5.0267e+2 -4.4490e+0 -1.0987e+1 -8.9874e+1 -1.3297e+0	-1.0262e+1±5.7148e+2 i -5.0267e+2 -8.9874e+1 -1.5214e+1±1.1622e+1 i -1.0987e+1 -2.9902e+0 -4.4490e+0

Optimal Characteristics:

$J^* = 2.0397e-1$

Eigenvalues = [-1.03888e+1±5.7148e+2i -5.0300e+2 -8.9955e+1 -5.1959e+1 -2.1141e+1 -9.6455e-1 -4.4576e+0 -1.1080e+1]

Table 4.5.5: Boiler Suboptimal Characteristics Based on Balancing, Aggregation, and Balagg for $R = 20 * I_m$

Suboptimal Balanced Characteristics	Suboptimal Aggregated Characteristics	Suboptimal Balagg Characteristics
Reduced order = 4		
$\mathcal{J}$=2.3666e-1 dJ=2.1680e+1% ϵ=4.4622e-1	$\mathcal{J}$=1.0578e+1 dJ=5.3386e+3% ϵ=1.4050e-2	$\mathcal{J}$=2.1567e+1 dJ=1.0989e+4% ϵ=4.6392e-3
Eigenvalues		
-7.7103e+2 -1.1162e+1±5.7122e+2 i -5.0460e+2 -9.0187e+1 -1.8997e+1 -9.3765e-1 -4.7560e+0 -1.0636e+1	-1.0987e+1 -8.9874e+1 -5.0267e+2 -2.5135e+1±5.7148e+2 i -1.5214e+1±1.1622e+1 i -1.8228e+1 -3.3988e+0	-2.6260e+1±5.7133e+2 i -5.0266e+2 -8.9874e+1 -6.8894e+1 -1.5214e+1±1.1622e+1 i -2.8185e+0 -1.0987e+1
Reduced order = 1		
$\mathcal{J}$=7.2611e+0 dJ=3.6333e+3% ϵ=1.9591e-2	$\mathcal{J}$=8.8964e+0 dJ=4.4741e+3% ϵ=1.4235e-2	$\mathcal{J}$=1.6642e+1 dJ=8.4565e+3% ϵ=7.6601e-3
Eigenvalues		
-1.0228e+1±5.7150e+2 i -5.0190e+2 -2.8989e+2 -9.0200e+1 -1.4058e+1 -1.0281e+1±2.4614e+0 i -4.1345e+0	-1.5214e+0±1.1622e+1 i -1.0262e+1±5.7148e+2 i -5.0267e+2 -4.4490e+0 -1.0987e+1 -8.9874e+1 -1.3742e+1	-1.0262e+1±5.7148e+2 i -5.0267e+2 -8.9874e+1 -4.0292e+1 -1.5214e+1±1.1622e+1 i -4.4490e+0 -1.0987e+1

Optimal Characteristics:

$J^* = 1.9450\text{e-}1$

Eigenvalues = [-8.3709e+2 -1.6872e+1±5.7128e+2i -4.6317e+2 -8.9727e+1 -1.9051e+1 -9.7097e-1 -4.4587e+0 -1.1097e+1]

Table 4.5.6: Boiler Suboptimal Characteristics Based on Balancing, Aggregation, and Balagg for $R=0.1*I_m$

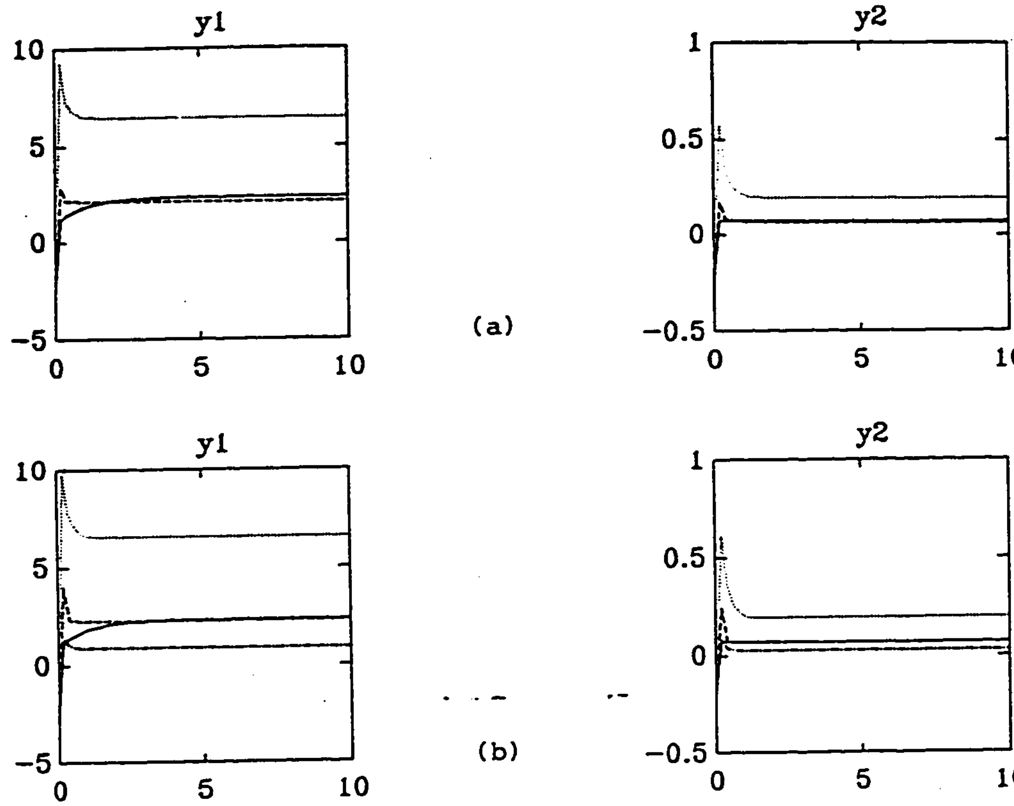

Fig.4.57 Closed Loop Responses: Optimal (–), Suboptimal Balanced (–.), Aggregated (..), and Balagg (-–); R= I. (a) r = 4 (b) r = 1

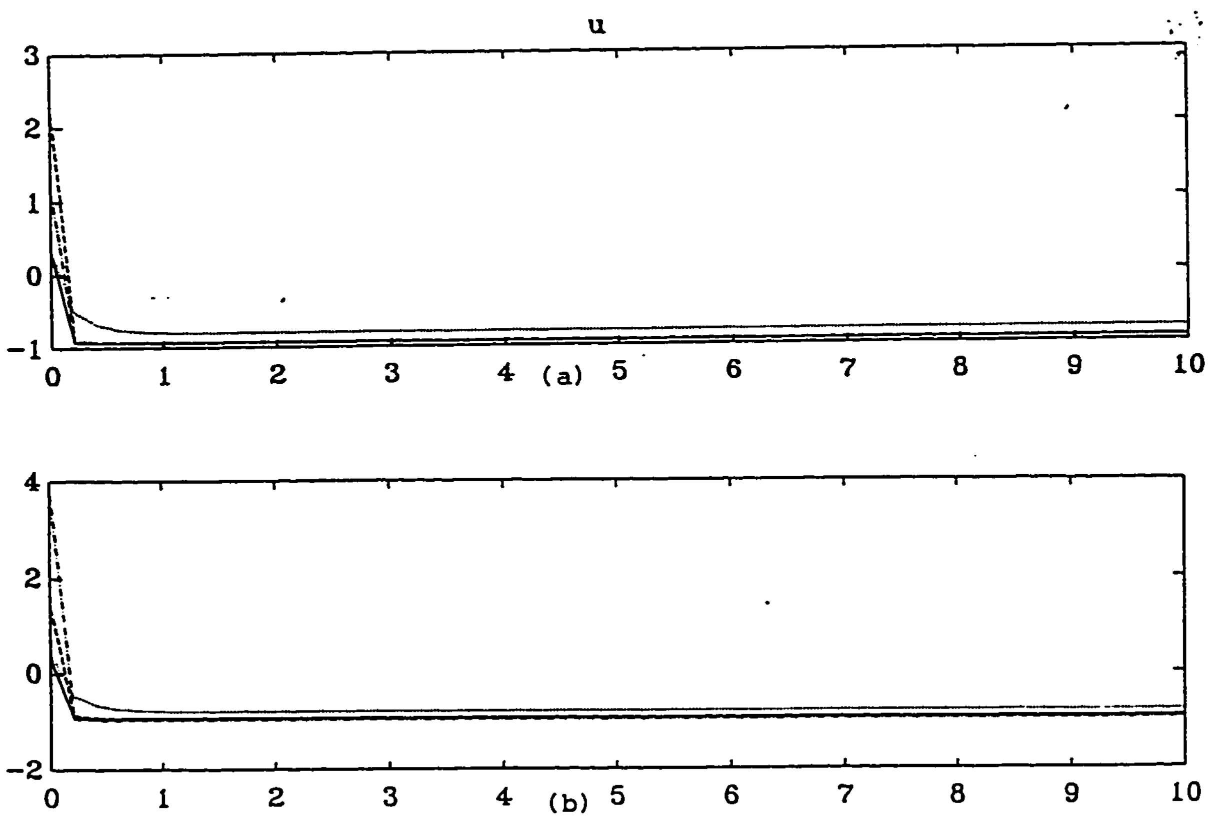

Fig.4.58 Closed Loop Controls: Optimal (–), Suboptimal Balanced (–.), Aggregated (..), and Balagg (-–); R= I. (a) r = 4.(b) r = 1

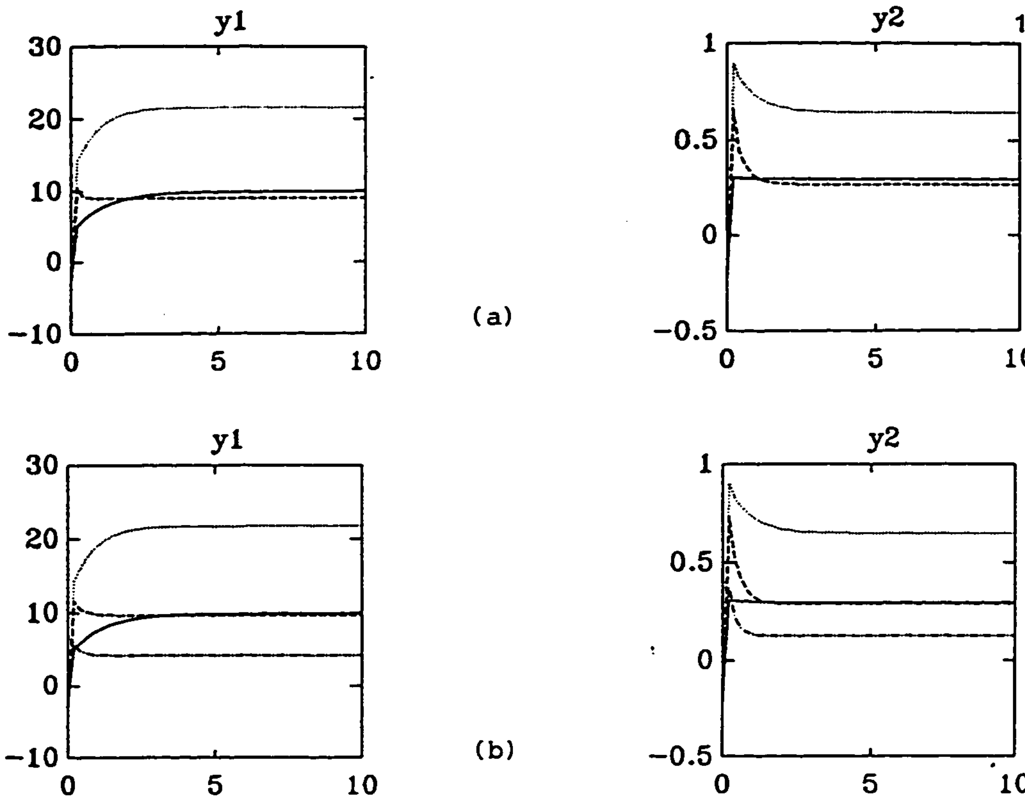

Fig 4.5.9 Closed Loop Responses: Optimal (–), Suboptimal Balanced (–.), Aggregated (..), and Balagg (––); R= 20I (a) r = 4. (b) r = 1

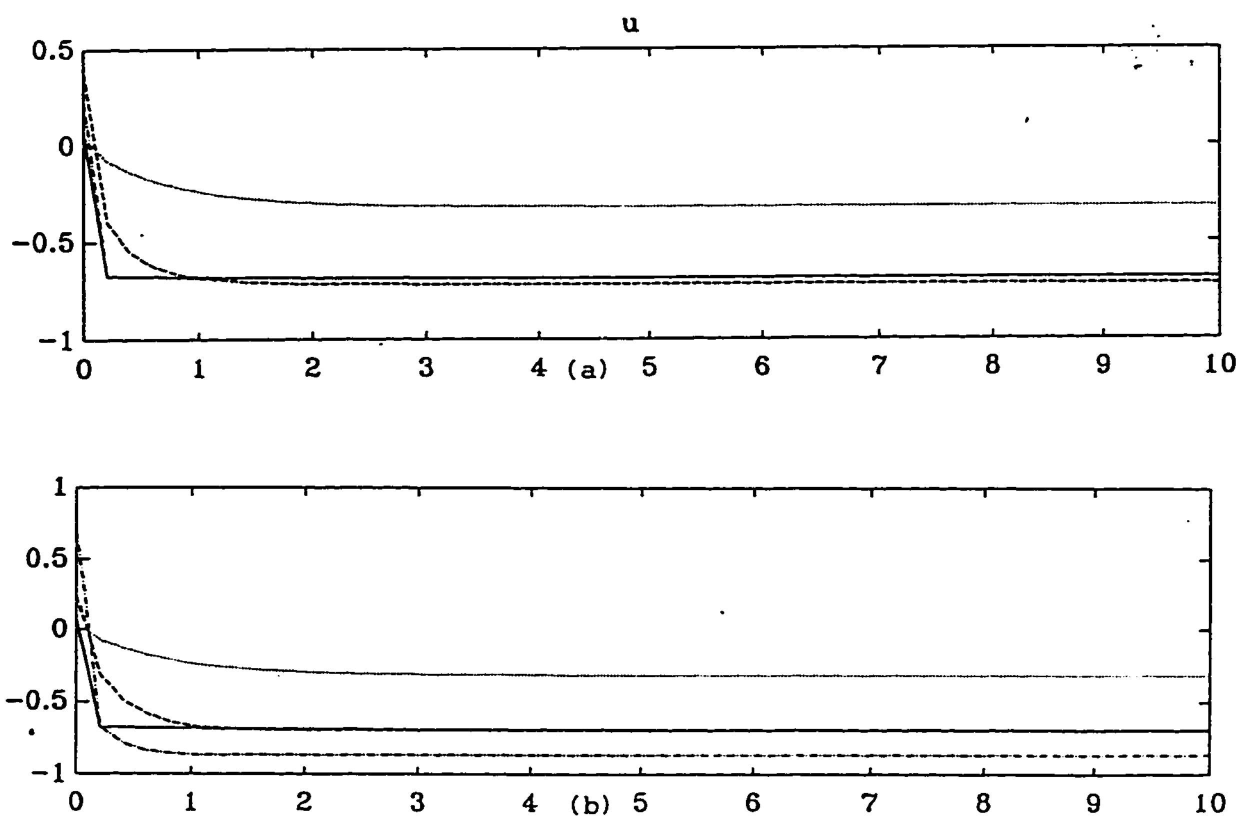

Fig. 4.5.10 Closed Loop Controls: Optimal (–), Suboptimal Balanced (–.), Aggregated (..), and Balagg (––); R= 20I (a) r = 4. (b) r =

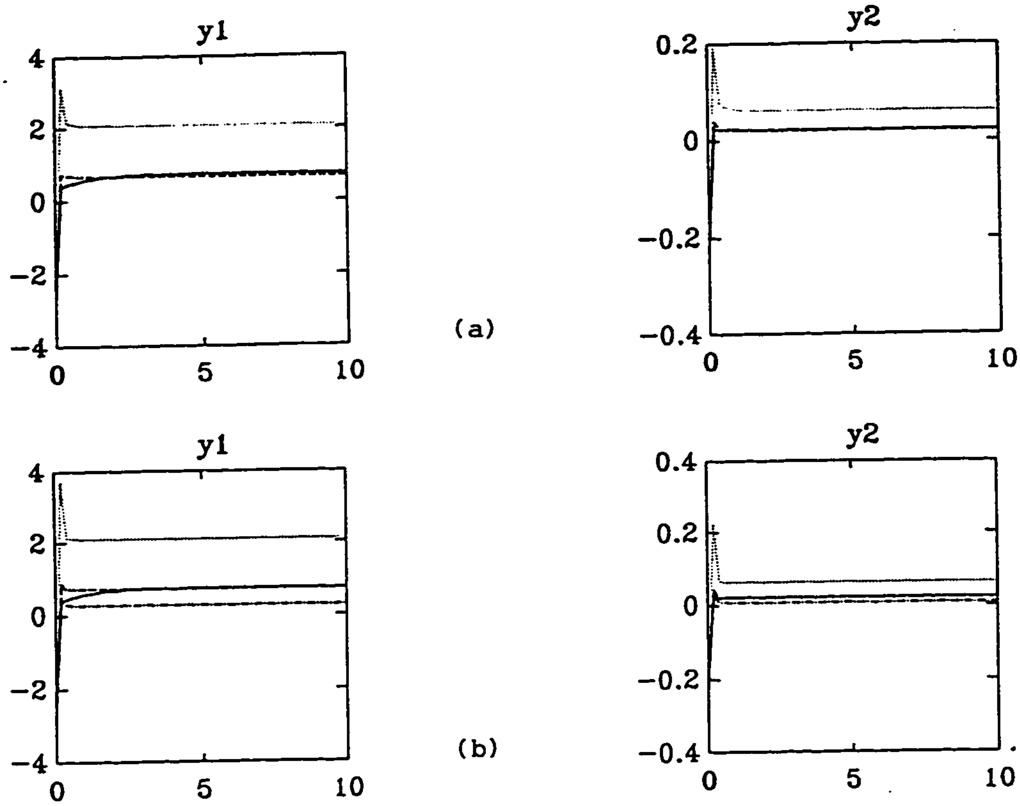

Fig.4.5.11 Closed Loop Responses: Optimal (–), Suboptimal Balanced (–.), Aggregated (..), and Balagg (––); R= 0.1I (a) r = 4 (b) r = 1

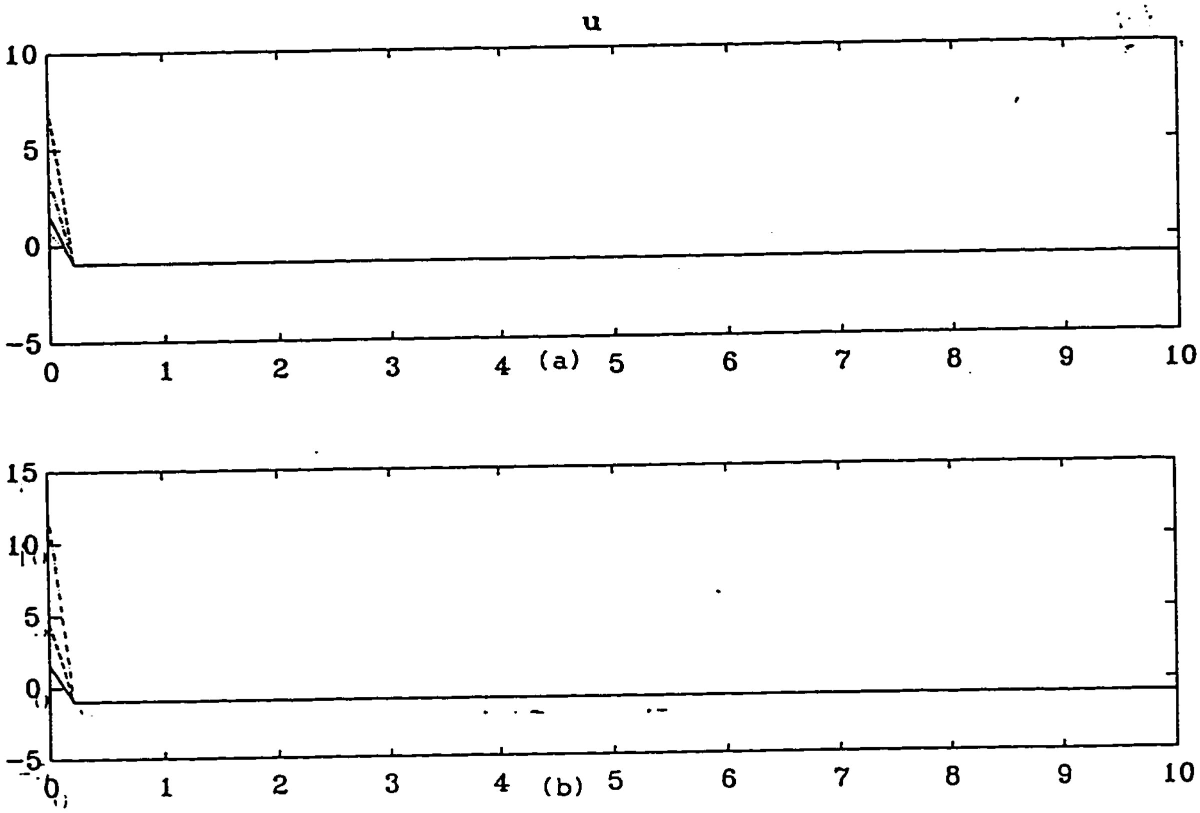

Fig. 4 .5.12 Closed Loop Controls: Optimal (–), Suboptimal Balanced (–.), Aggregated (..), and Balagg (––); R= 0.1I (a) r = 4 (b) r = 1

4.6 Comparison of Discrete Suboptimal Control Laws Based on Balancing and Aggregation: Simulation and Results

In the discrete balancing case, it is possible to derive a suboptimal control law based on the reduced order model. The analysis is completely analogous to the continuous time case. Therefore, it will not be repeated here.

In the following, we are going to simulate an eighth order example. For balancing, the suboptimal control will be based on two versions of reduced order models. The first version is the one developed by Pernebo and Silverman [26] which we will refer to as balancing and it is as given in chapter 2. It is the discrete version of (4.2.5). This reduced order model is known to be unbalanced (theorem 2.5). The other version is what is called by Fernando and Nickolson [36] as Singular Perturbation Approximation of balanced system. It is obtained by setting $\alpha = -1$ in the reduced order model developed in [37] and given in (2.4.25). We are going to refer to this model, which is known to be balanced (theorem 2.6), as 2-time scale balancing. For aggregation, the discrete version of Davison's model will be used. Discrete versions of suboptimal control laws used in section 4.5 will be applied to the corresponding reduced order models of this section.

4.6.1 Chemical Reactor

This is an example of an eighth order discrete chemical reactor considered in [77]. The A, B, and H matrices of the system are given below.

$$A = \begin{bmatrix} 0.5623 & -0.01642 & 0.01287 & -0.01610 & 0.02094 & -0.02988 & 0.01830 & 0.008743 \\ 0.1020 & 0.6114 & -0.02468 & 0.02468 & -0.03005 & 0.04195 & -0.02559 & 0.03889 \\ 0.1361 & 0.2523 & 0.6410 & -0.03404 & 0.03292 & -0.04296 & 0.02588 & 0.08467 \\ 0.09951 & 0.2859 & 0.3476 & 0.6457 & -0.03249 & 0.03316 & -0.01913 & 0.1103 \\ -0.04794 & 0.08708 & 0.3297 & 0.3102 & 0.6201 & -0.03015 & 0.01547 & 0.08457 \\ -0.1373 & -0.1224 & 0.1705 & 0.3106 & 0.1910 & 0.5815 & -0.01274 & 0.05394 \\ -0.1497 & -0.1692 & 0.1165 & 0.2962 & 0.1979 & 0.07631 & 0.5242 & 0.04702 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.6065 \end{bmatrix} \tag{4.6.1a}$$

$$B' = [-0.1774 \quad -0.2156 \quad -0.2194 \quad -0.09543 \quad 0.05790 \quad 0.09303 \quad 0.08962 \quad 0] \tag{4.6.1b}$$

$$H = [-0.0049 \quad 0.0049 \quad -0.006 \quad 0.01 \quad 0.02630 \quad 0.3416 \quad 0.6759 \quad 0] \tag{4.6.1c}$$

Also, the initial condition vector x_0 and the weighting matrix Q are

$$x_0' = [0.5 \quad 0.5 \quad 0.5 \quad 0.5 \quad 0.5 \quad 0.5 \quad 0.5 \quad 0.5] \tag{4.6.1d}$$

$$Q = I_n, \ where \ n = 8 \tag{4.6.2e}$$

The open loop eigenvalues eva and the singular values of the balanced system Σ are:

eva = [6.7322e-1±1.7791e-1i 5.1456e-1±6.6335e-2i 5.9747e-1±5.5735e-2i 6.1570e-1 6.0650e-1]

Σ = [1.6128e+0 8.0119e-1 1.5063e-1 2.4812e-3 7.1005e-6 3.6561e-8 9.1467e-10 2.0197e-13]

Three cases will be considered : $R=I_m$, $R=20*I_m$ and $R=0.1*I_m$. In every case the full order system will be reduced to a third and a first order system using balancing, 2-time scale balancing, aggregation, and balagg. Then, the suboptimal control laws obtained using the reduced order models will be applied to the full order system and the performance characteristics will be evaluated.

4.6.1.1 Simulation and Results

Tables (4.6.1-4.6.3), give a summary of the different characteristics of the optimal and the suboptimal closed loop systems using balancing, 2-time scale balancing, aggregation and balagg. Comparing the characteristics of the suboptimal closed loop systems obtained by balancing, 2-time scale balancing, aggregation and balagg to those of the optimal closed loop system, we see the following:

1. Stability: All the methods give stable suboptimal closed loop systems.

2. Performance degradation dJ: In general, it is very small for balancing and 2-time scale balancing for $r=3$ and small for $r=1$. However, it is not as small for aggregation nor balagg although it is smaller for balagg than aggregation.

3. Suboptimality degree ϵ: It is very close to unity for balancing and 2-time scale balancing when $r=3$ and it is acceptable when $r=1$. For aggregation, in general, it is

acceptable but for balagg it is small except for $R=20*I_m$ where it becomes large. Also, as in the continuous time case, ϵ improves for balagg when r changes from 3 to 1.

4. Eigenvalues: Suboptimal eigenvalues based on balancing and 2-time scale balancing, in general, are very close to the optimal ones for $r=3$ and far for $r=1$ except for $R=20*I_m$ where they become close. However, they are far for aggregation and balagg except for $r=3$ and $R=I_m$ and $R=20*I_m$

5. Suboptimal step responses and controls: Figures (4.6.1-4.6.6) show the suboptimal responses and controls obtained by the different reduction methods. When $r=3$, suboptimal responses and controls based on balancing and 2-time scale balancing are identical to the optimal ones for all R. Also, those based on aggregation and balagg are far optimum but they are closer for balagg. When $r=1$, suboptimal responses and controls based on aggregation and balagg remained far from optimum. Although, suboptimal responses based on balancing and 2-time scale balancing shifted from their previous values (except for balancing when $R=I_m$), and became not identical to optimum, they stayed close to the optimal responses. However, the controls shifted from optimality.

Suboptimal Balanced Characteristics	Suboptimal 2-time Scale Balanced Characteristics	Suboptimal Aggregated Characteristics	Suboptimal Balagg Characteristics
Reduced order = 3			
J=1.2978e+1 dJ=1.0259e-1% ϵ=9.7889e-1	J=1.2978e+1 dJ=1.0747e-1% ϵ=9.7863e-1	J=1.6510e+1 dJ=2.7346e+1% ϵ=6.7067e-1	J=1.3020e+1 dJ=4.3210e-1% ϵ=2.7085e-1
Eigenvalues			
5.1741e-1± 2.8691e-1 i 5.0491e-1± 6.1396e-2 i 5.4650e-1± 3.1372e-2 i 6.0998e-1 6.0741e-1	5.1894e-1± 2.8507e-1 i 5.0509e-1± 6.1684e-2 i 5.4705e-1± 3.2028e-2 i 6.0314e-1 6.0982e-1	5.8408e-1± 1.9005e-1 i 5.1456e-1± 6.6335e-2 i 5.9747e-1± 5.5735e-2 i 6.1610e-1 6.0650e-1	5.0672e-1± 2.1716e-1 i 5.1456e-1± 6.6335e-2 i 5.9747e-1± 5.5735e-2 i 5.7857e-1 6.0650e-1
Reduced order = 1			
J=1.5439e+1 dJ=1.9090e+1% ϵ=5.9965e-1	J=1.4998e+1 dJ=1.5687e+1% ϵ=5.7615e-1	J=1.9875e+1 dJ=5.3305e+1% ϵ=4.5994e-1	J=1.8673e+1 dJ=4.4035e+1% ϵ=5.1600e-1
Eigenvalues			
7.1967e-1± 4.0837e-1 i 4.7044e-1± 9.7104e-2 i 5.2038e-1± 4.6762e-2 i 5.4559e-1 6.0975e-1	7.3079e-1± 3.7315e-1 i 4.8421e-1± 1.0214e-1 i 5.2217e-1± 4.9755e-2 i 5.4908e-1 6.0975e-1	6.7322e-1+ 1.7791e-1 i 5.9548e-1- 1.5736e-1 i 5.1456e-1± 6.6335e-2 i 5.9747e-1± 5.5735e-2 i 6.1570e-1 6.0650e-1	6.7322e-1+ 1.7791e-1 i 5.6084e-1- 1.4821e-1 i 5.1456e-1± 6.6335e-2 i 5.9747e-1± 5.5735e-2 i 6.1570e-1 6.0650e-1

Optimal Characteristics:

$J^* = 1.2964e+1$

Eigenvalues = [5.1085e-1±2.8557e-1i 5.1003e-1±6.0354e-2i 5.5294e-1±3.1867e-2i 5.9578e-1 6.1014e-1]

Table 4.6.1: Chemical Reactor Suboptimal Characteristics Based on Balancing, 2-time Scale Balanced , Aggregation, and Balagg for $R = I_m$

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Suboptimal Balanced Characteristics	Suboptimal 2-time Scale Balanced Characteristics	Suboptimal Aggregated Characteristics	Suboptimal Balagg Characteristics
Reduced order = 3			
J=2.1014e+1 dJ=6.8218e-3% ϵ=9.9811e-1	J=2.1014e+1 dJ=6.8218e-3% ϵ=9.9812e-1	J=2.2793e+1 dJ=8.4710e+0% ϵ=8.7144e-1	J=2.1516e+1 dJ=2.3970e+0% ϵ=8.2610e-1
Eigenvalues			
6.5368e-1± 2.1026e-1 i 5.1412e-1± 6.6380e-2 i 5.8724e-1± 5.4648e-2 i 6.0415e-1 6.1184e-1	6.5384e-1± 2.1030e-1 i 5.1413e-1± 6.6383e-2 i 5.8725e-1± 5.4799e-2 i 6.0396e-1 6.1176e-1	6.6633e-1± 1.7983e-1 i 5.1456e-1± 6.6335e-2 i 5.9747e-1± 5.5735e-2 i 6.1573e-1 6.0650e-1	6.6269e-1± 1.8562e-1 i 5.1456e-1± 6.6335e-2 i 5.9747e-1± 5.5735e-2 i 6.1087e-1 6.0650e-1
Reduced order = 1			
J=2.1739e+1 dJ=3.4568e+0% ϵ=8.7456e-1	J=2.1495e+1 dJ=2.2972e+0% ϵ=8.8316e-1	J=2.3321e+1 dJ=1.0986e+1% ϵ=8.3230e-1	J=2.3170e+1 dJ=1.0265e+1% ϵ=8.4520e-1
Eigenvalues			
7.2602e-1± 2.4197e-1 i 5.4552e-1± 9.3563e-2 i 5.2062e-1± 6.6309e-2 i 5.6670e-1 6.0980e-1	7.1720e-1± 2.2113e-1 i 5.6022e-1± 9.0388e-2 i 5.1793e-1± 6.6821e-2 i 5.7171e-1 6.0984e-1	6.7322e-1+ 1.7791e-1 i 6.6841e-1- 1.7664e-1 i 5.1456e-1± 6.6335e-2 i 5.9747e-1± 5.5735e-2 i 6.1570e-1 6.0650e-1	6.7322e-1+ 1.7791e-1 i 6.6558e-1- 1.7589e-1 i 5.1456e-1± 6.6335e-2 i 5.9747e-1± 5.5735e-2 i 6.1570e-1 6.0650e-1

Optimal Characteristics:

$J^* = 2.1013e+1$

Eigenvalues = [6.5274e-1±2.1025e-1i 5.1414e-1±6.6129e-2i 5.8755e-1±5.3286e-2i 6.1241e-1 6.0467e-1]

Table 4.6.2: Chemical Reactor Suboptimal Characteristics Based on Balancing, 2-time Scale Balanced , Aggregation, and Balagg for $R = 20 * I_m$

Suboptimal Balanced Characteristics	Suboptimal 2-time Scale Balanced Characteristics	Suboptimal Aggregated Characteristics	Suboptimal Balagg Characteristics
Reduced order = 3			
J=9.2826e+0 dJ=2.7235e-1% ϵ=9.5538e-1	J=9.2835e+0 dJ=2.8132e-1% ϵ=9.5407e-1	J=1.1185e+1 dJ=2.0812e+1% ϵ=7.2648e-1	J=1.0705e+1 dJ=1.5639e+1% ϵ=1.2538e-1
Eigenvalues			
3.6195e-1± 2.0669e-1 i 3.6728e-1 5.1626e-1± 5.0598e-2 i 5.5371e-1 6.0938e-1± 9.9650e-4 i	3.7377e-1± 1.9791e-1 i 3.4630e-1 5.1647e-1± 4.1341e-2 i 5.5557e-1 6.0403e-1 6.0953e-1	4.0007e-1± 9.5393e-2 i 5.1456e-1± 6.6335e-2 i 5.9747e-1± 5.5735e-2 i 6.1659e-1 6.0650e-1	3.8416e-1± 1.0081e-1 i 5.1456e-1± 6.6335e-2 i 5.9747e-1± 5.5735e-2 i 5.2491e-1 6.0650e-1
Reduced order = 1			
J=1.3233e+1 dJ=4.2942e+1% ϵ=5.5914e-1	J=1.2449e+1 dJ=3.4481e+1% ϵ=4.4925e-1	J=1.3497e+1 dJ=4.5799e+1 ϵ=4.0570e-1	J=1.2610e+1 dJ=3.6212e+1% ϵ=4.1942e-1
Eigenvalues			
6.2562e-1± 5.2125e-1 i 3.5035e-1 4.7633e-1 5.0829e-1± 3.6041e-2 i 5.3008e-1 6.0974e-1	6.8970e-1± 4.6259e-1 i 4.4704e-1± 7.8498e-2 i 5.1689e-1± 4.2284e-2 i 5.3996e-1 6.0975e-1	3.3892e-1- 8.9564e-2 i 6.7322e-1+ 1.7791e-1 i 5.1456e-1± 6.6335e-2 i 5.9747e-1± 5.5735e-2 i 6.1570e-1 6.0650e-1	2.7102e-1- 7.1619e-2 i 6.7322e-1+ 1.7791e-1 i 5.1456e-1± 6.6335e-2 i 5.9747e-1± 5.5735e-2 i 6.1570e-1 6.0650e-1

Optimal Characteristics:

$J^* = 9.2574e+0$

Eigenvalues = [3.2630e-1±2.1350e-1i 4.3202e-1 5.1959e-1±5.4314e-2i 5.6533e-1 5.9250e-1 6.1001e-1]

Table 4.6.3: Chemical Reactor Suboptimal Characteristics Based on Balancing, 2-time Scale Balanced , Aggregation, and Balagg for $R = 0.10 * I_m$

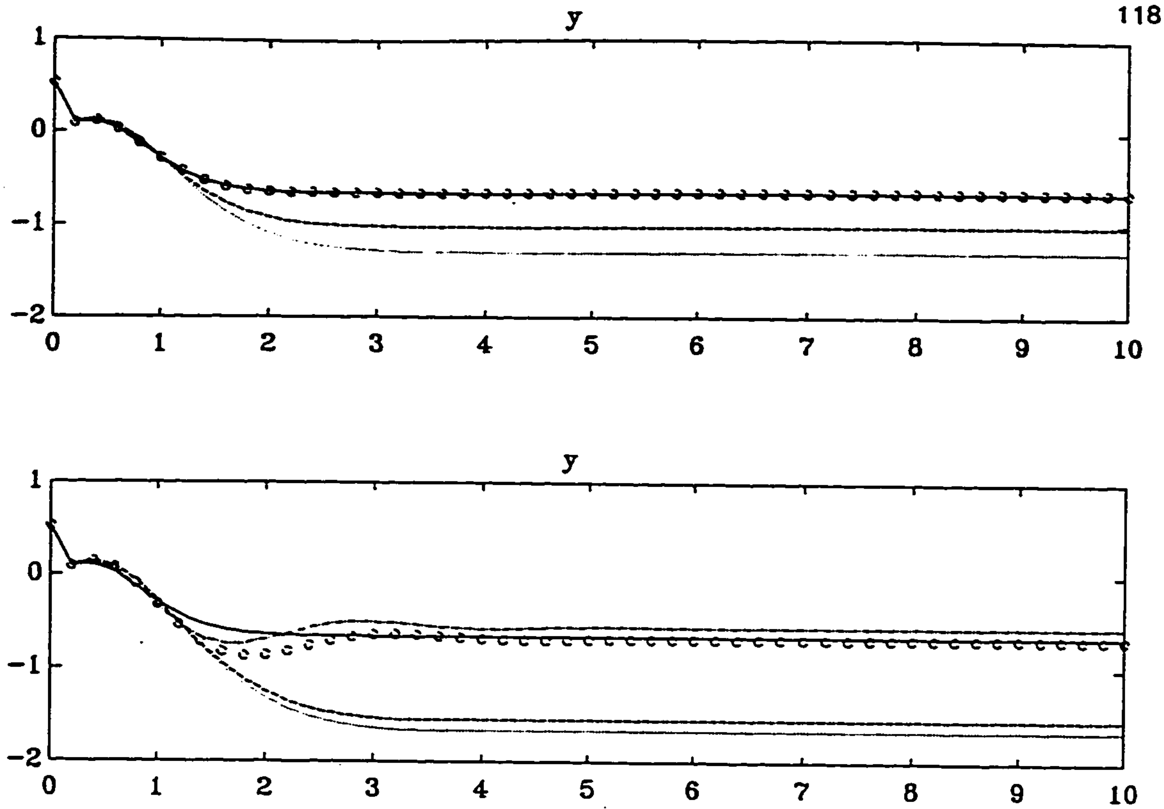


Fig.4.6.1 Closed Loop Responses: Optimal (–), Suboptimal Balanced (–.), 2–time Scale Bal. (o), Aggregated (..), and Balagg (––); R=I, (a) r=3, (b) r=1

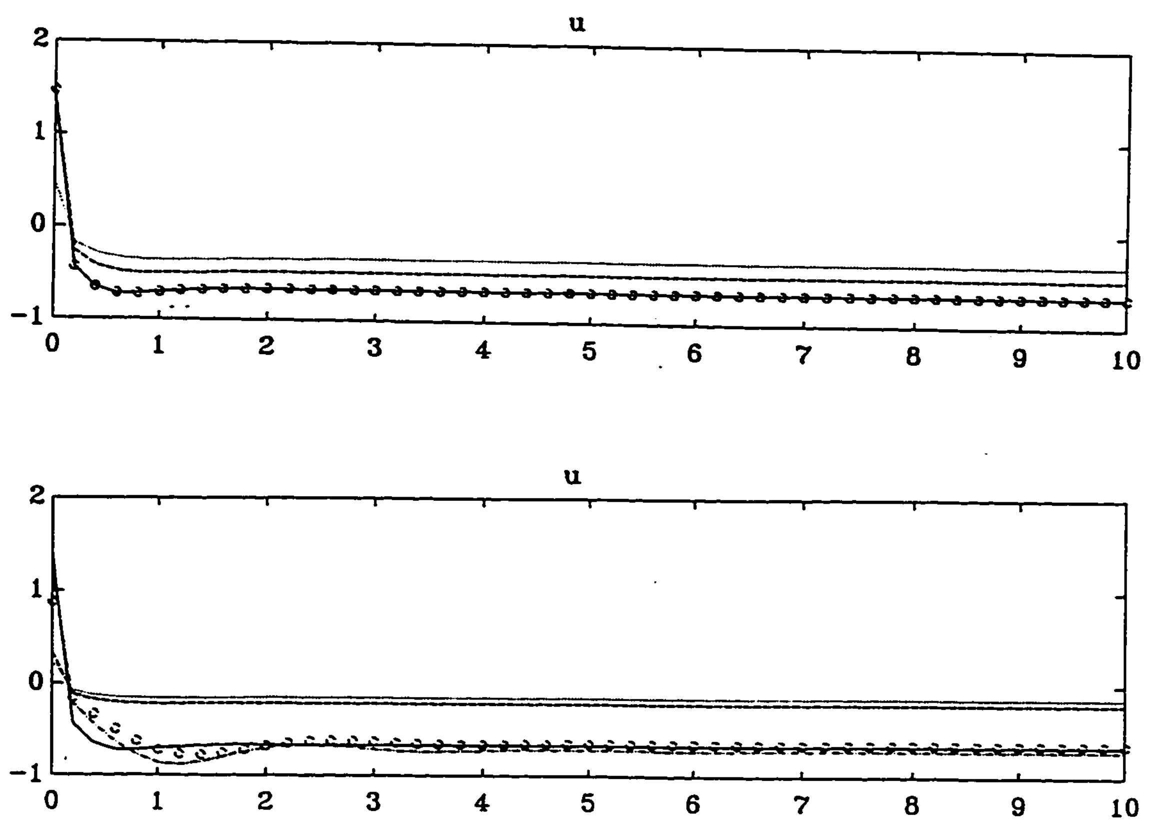


Fig.4.6.2 Closed Loop Controls: Optimal (–), Suboptimal Balanced (–.), 2–time Scale Bal. (o), Aggregated (..) and Balagg (––); R=I, (a) r=3 (b) r=1

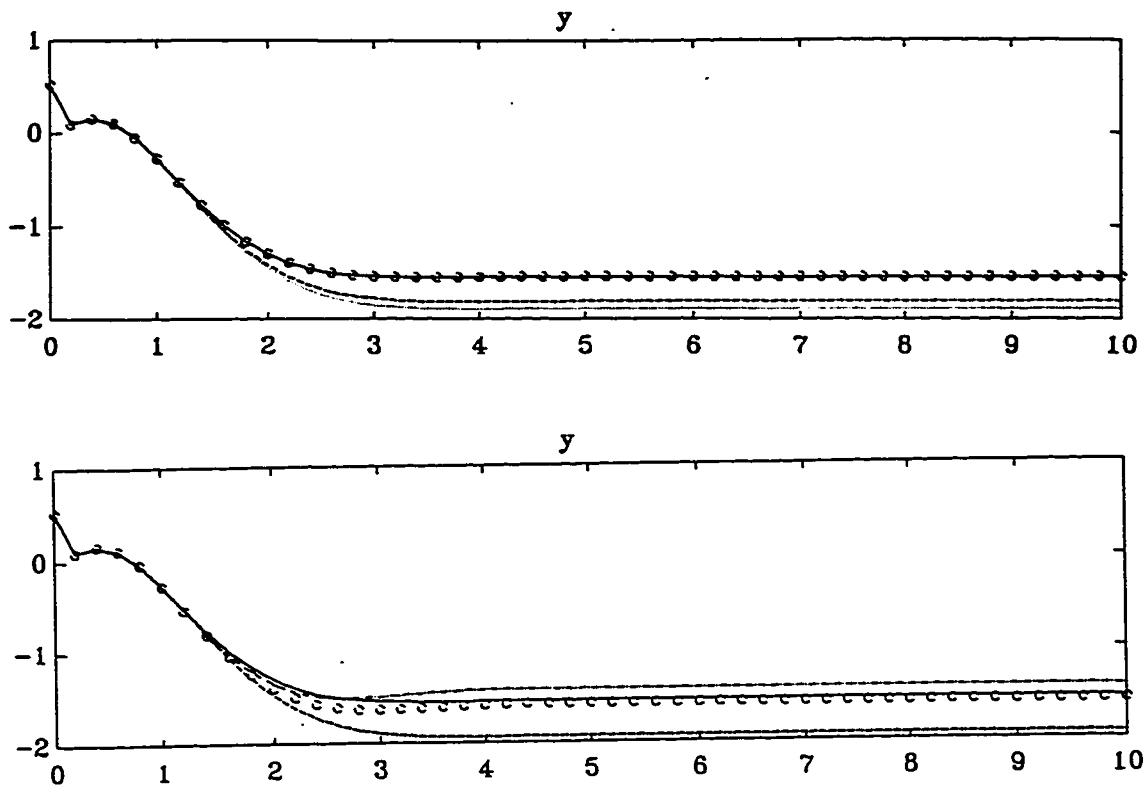


Fig.4.6.3 Closed Loop Responses: Optimal (−), Suboptimal Balanced (−.), 2-time Scale Bal. (o), Aggregated (..), and Balagg (−−); R=20I, (a) r=3, (b) r=1

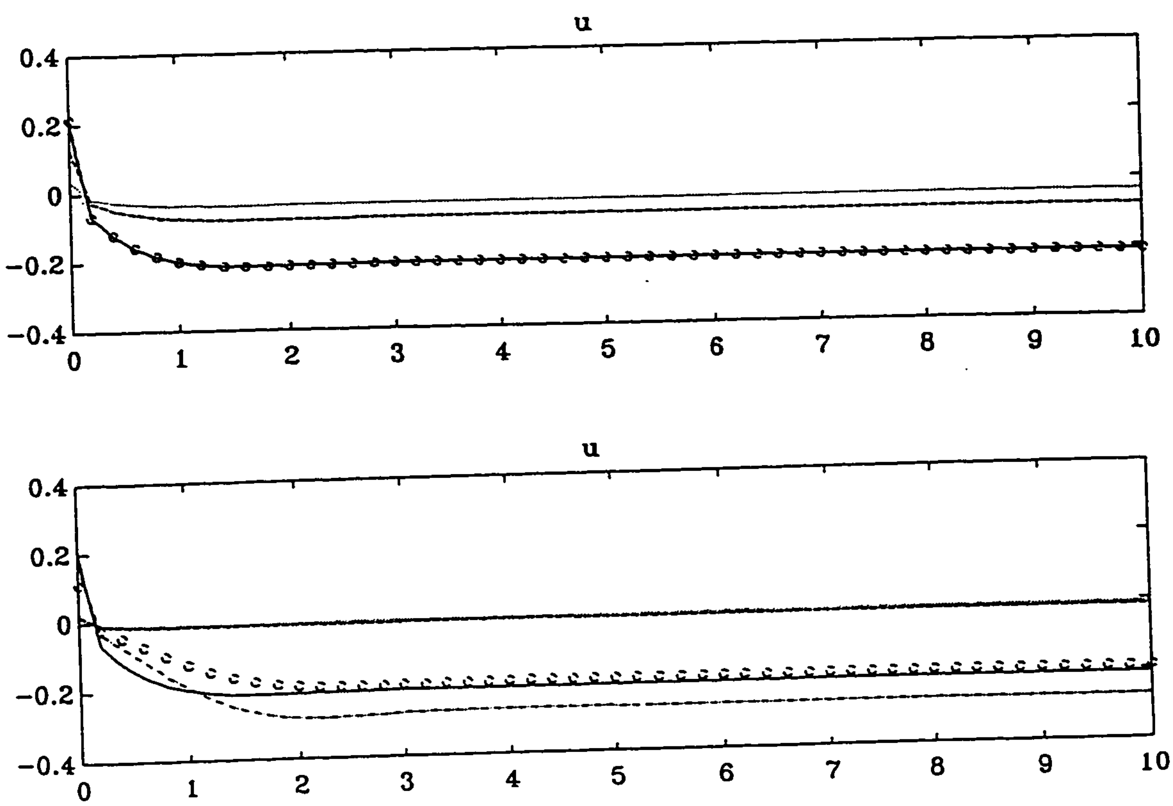


Fig.4.6.4 Closed Loop Controls: Optimal (−), Suboptimal Balanced (−.), 2-time Scale Bal. (o), Aggregated (..) and Balagg (−−); R=20I, (a) r=3 (b) r=1

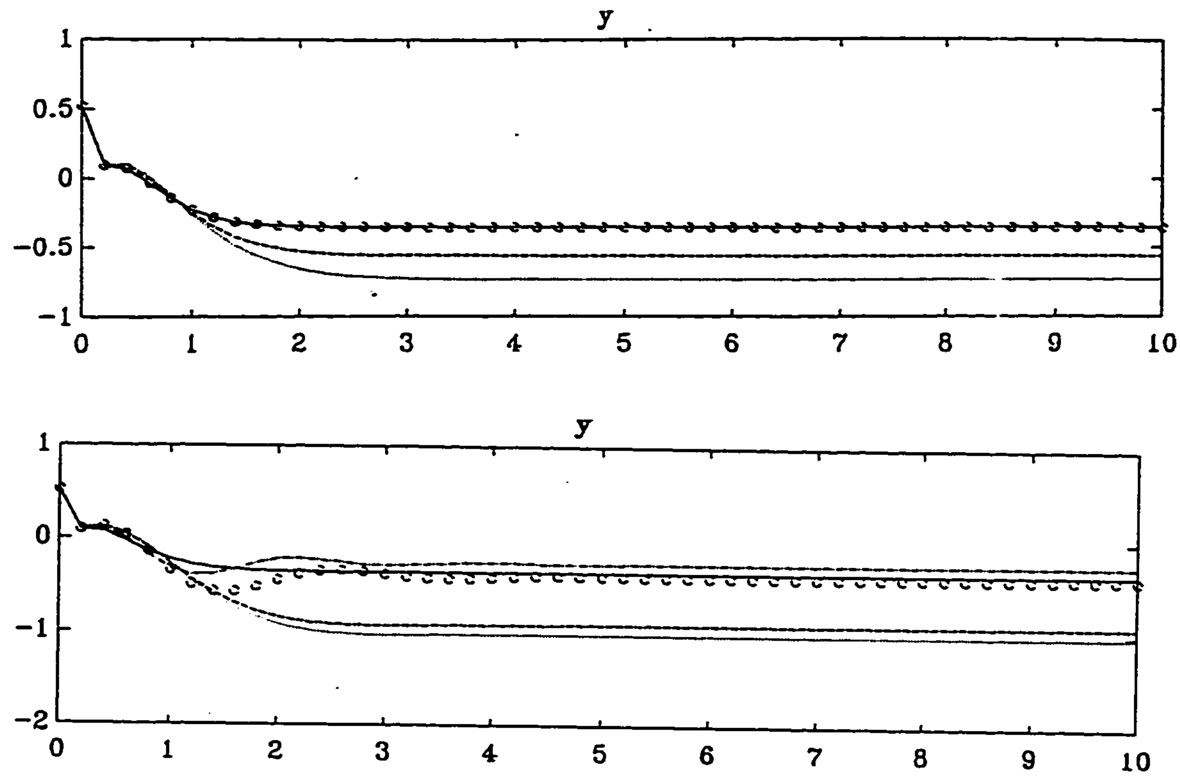


Fig.4.6.5 Closed Loop Responses: Optimal (–), Suboptimal Balanced (–.), 2–time Scale Bal. (o), Aggregated (..), and Balagg (––); R=0.1I, (a) r=3, (b) r=1

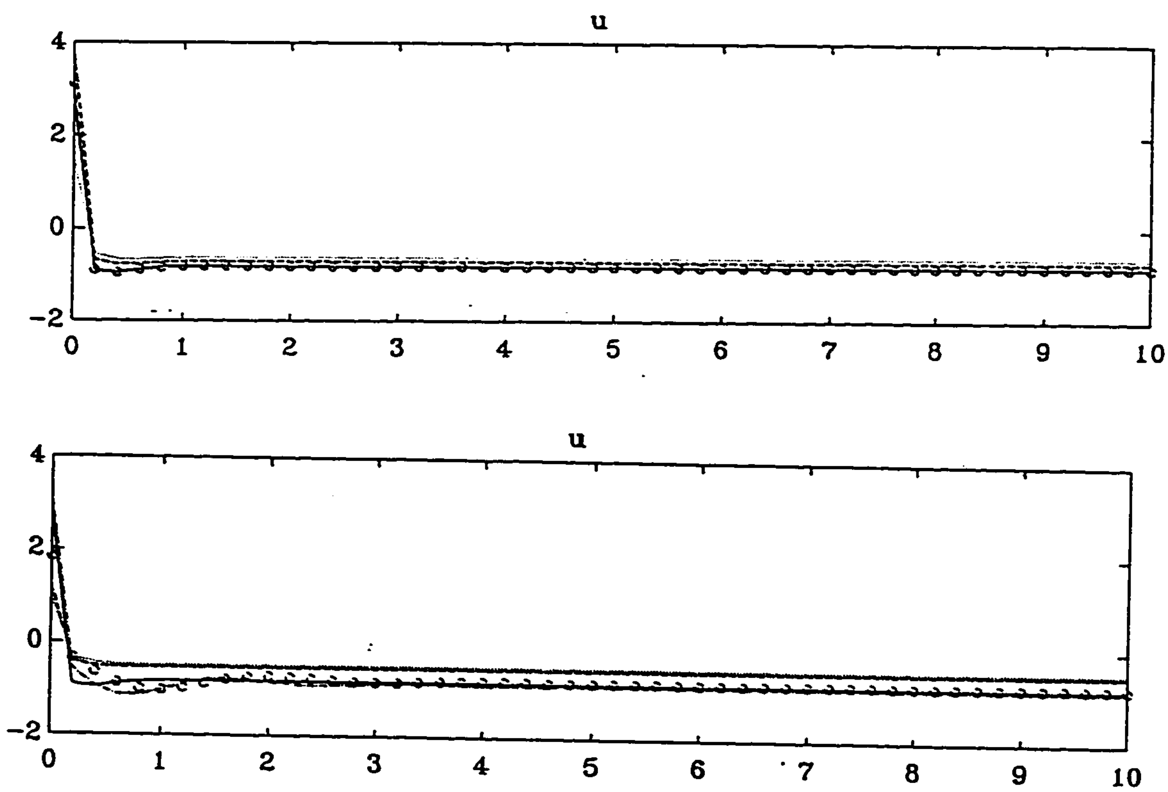


Fig.4.6.6 Closed Loop Controls: Optimal (–), Suboptimal Balanced (–.), 2–time Scale Bal. (o), Aggregated (..) and Balagg (––); R=0.1I, (a) r=3 (b) r=1

4.7 Comparison of Continuous Suboptimal Control Laws Based on Balancing and Singular Perturbation: Simulation and Results

In the following, we are going to simulate a fifth order voltage regulator plant [57] which is in the singularly perturbed form given by (3.2.37). For balancing, two versions of reduced order models will be considered. The first one is the one developed by Moore [24] which we are going to refer to as balancing and given in (4.2.5). This reduced order model is known to be balanced. The other version is the 2-time scale approximation (or Singular Perturbation Approximation as called in [35]) of the balanced system. It is given in (2.3.4) in connection with theorem 2.4. The suboptimal control law (4.2.15) will be applied to both reduced order models. For singular perturbation, the suboptimal control law (3.2.44) will be applied to the slow subsystem of the unbalanced singularly perturbed system given in (3.2.38)

4.7.1 Voltage Regulator

The system matrices of the singularly perturbed voltage regulator which is of the form (3.2.37) are given below.

$$A_{11} = \begin{bmatrix} -0.2 & 0.5 \\ 0 & -0.5 \end{bmatrix} \qquad A_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 1.6 & 0 & 0 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad A_{22} = \begin{bmatrix} -1.428 & 8.714 & 0 \\ 0 & -2.5 & 7.5 \\ 0 & 0 & -1 \end{bmatrix}$$

$$(4.7.1a)$$

$$B_1' = [0 \quad 0] \; , \; B_2' = [0 \quad 0 \quad 3] \qquad (4.7.1b)$$

$$C_1 = [1 \quad 0 \quad 0 \quad 0 \quad 0] \; , \; C_2 = [0 \quad 1 \quad 0 \quad 0 \quad 1] \qquad (4.7.1c)$$

Also, the initial condition vector x_0 is

$$x_0' = [0.5 \quad 0.5 \quad 0.5 \quad 0.5 \quad 0.5] \qquad (4.7.1d)$$

The open loop eigenvalues *eva* are

$$eva = [-1.43e{+}1 \;\; -1.00e{+}1 \;\; -2.00e{-}1 \;\; -5.00e{-}1 \;\; -2.50e{+}1]$$

From the open loop eigenvalues, it is possible to see that the system can be modeled by two subsystems: slow subsystem of order $r=2$ and fast subsystem of order $r=3$. Three cases will be considered : $R=I_m$, $R=100*I_m$ both with $\mu=0.1$, and $R=I_m$ with $\mu=0.02$. In every case the full order system will be reduced to a second order system using balancing, 2-time scale balancing, and singular perturbation. Then, the suboptimal control laws obtained using the reduced order models will be applied to the full order system and the performance characteristics will be evaluated.

The singular values of the balanced system at the different μ are

$$\Sigma_{\mu=0.1} = [2.6909e{+}2 \;\; 6.8782e{+}1 \;\; 6.9428e{+}0 \;\; 8.1415e{-}1 \;\; 3.1620e{-}2]$$

$$\Sigma_{\mu=0.02} = [2.6422e{+}2 \;\; 6.3669e{+}1 \;\; 6.3059e{-}1 \;\; 3.4416e{-}1 \;\; 6.1344e{-}3]$$

4.7.1.1 Simulation and Results

Table (4.7.1), gives a summary of the different characteristics of the optimal and the suboptimal closed loop systems using balancing, 2-time scale balancing, and singular perturbation. Comparing the characteristics of the suboptimal closed loop system obtained by balancing, 2-time scale balancing, and singular perturbation to those of the optimal closed loop system, we see the following:

1. Stability: All the methods give stable suboptimal closed loop systems except for the closed loop system based on singular perturbation when $R = I_m$ and $\mu = 0.1$.

2. Performance degradation dJ: It is, in general, very small for balancing and 2-time scale balancing while it is not small for singular perturbation. In fact, it is infinity when the system is unstable.

3. Suboptimality degree ϵ: For $R = I_m$ and $\mu = 0.1$, it is very small for balancing and 2-time scale balancing while it is not applicable for singular perturbation. For $R = 100 * I_m$ and $\mu = 0.1$, it is good for balancing and 2-time scale balancing while it is very small for singular perturbation. For $R = I_m$ and $\mu = 0.02$, it is very small for balancing and 2-time scale balancing while it is extremely small for singular perturbation

4. Closed loop eigenvalues: It can be seen that all the methods resulted in eigenvalues far from optimum. Also, comparing the eigenvalues obtained by balancing to those obtained by 2-time scale balancing, it can be seen that they are almost identical.

5. Suboptimal step responses and controls: Figures (4.7.1-4.7.4) show the responses and the controls. Suboptimal responses and controls based on balancing and 2-time scale balancing are identical to the optimum ones for all the cases. Suboptimal responses and controls based on singular perturbation are far from optimum except for the control when $R = I_m$ and $\mu = 0.02$. Also, as seen from fig. 4.7.1a and 4.7.2a, singular perturbation has resulted in unstable system when $R = I_m$ and $\mu = 0.1$.

Suboptimal Balanced Characteristics	Suboptimal 2-time Scale Balanced Characteristics	Suboptimal Singular Perturbation Characteristics
$R = I_m$; $\mu=0.1$		
J=4.9483e-1 dJ=1.1530e+1% ϵ=8.3963e-2	J=4.9405e-1 dJ=1.1355e+1% ϵ=8.1922e-2	$J=\infty$ $dJ=\infty$ ϵ=N/A
Eigenvalues		
-7.0144e+1 -6.5141e-1 -2.5294e+1 -1.2072e+1±1.1961e+0 i	-7.3250e+1 -6.1300e-1 -1.2071e+1±1.4449e+0 i -2.5318e+1	-2.6414e+1±1.0804e+1 i 1.6793e+0±1.0911e+1 i -5.3063e-1
$R = 100*I_m$; $\mu=0.1$		
J=1.0939e+0 dJ=1.4189e+0% ϵ=6.6688e-1	J=1.0906e+0 dJ=1.1104e+0% ϵ=6.5215e-1	J=1.27225e+0 dJ=5.9691e+1% ϵ=4.6615e-2
Eigenvalues		
-1.5215e+1 -8.1097e+0±1.8522e+0 i -2.4918e+1 -6.4802e-1	-1.5338e+1 -8.1972±2.1344e+0 i -2.4906e+1 -6.1321e-1	-2.3617e+1±6.7588e+0 i -1.1177e+0±6.9700e+0 i -5.3028e-1
$R = I_m$; $\mu=0.02$		
J=3.7410e-1 dJ=2.3816e-1% ϵ=6.7521e-2	J=3.7411e-1 dJ=2.4093e-1% ϵ=6.7435e-2	J=3.8857e-1 dJ=4.1169e+0% ϵ=3.7361e-3
Eigenvalues		
-5.3572e-1 -9.2583e+1 -6.1536e+1 -5.4953e+1 -1.2427e+2	-5.3409e-1 -9.2766e+1 -6.1552e+1 -5.4958e+1 -1.2426e+2	-1.1449e+2±2.7409e+1 i -8.8418e+0±2.8582e+1 i -5.3066e-1

TABLE 4.7.1: Voltage Regulator Suboptimal Characteristics Based on Balancing, 2-time Scale Balancing, and Singular Perturbation

Optimal Characteristics:

μ = 0.1

$R = I_m$: J^* =4.4367e-1
Eigenvalues = [-3.5080e+1±1.0655e+1i -3.9626e+0±1.4765e+1i -5.3019e-1]

$R = 100*I_m$: J^* =1.0786e+0
Eigenvalues = [-2.3904e+1 -6.1091e+0±5.6779e+0i -18893e+1 -5.3028e-1]

μ = 0.02
$R = I_m$: J^* = 3.7321e-1
Eigenvalues = [-1.5523e+2±3.0161e+1i -2.5022e+1±3.0710e+1i -5.3058e-1]

TABLE 4.7.1: Continue

0083

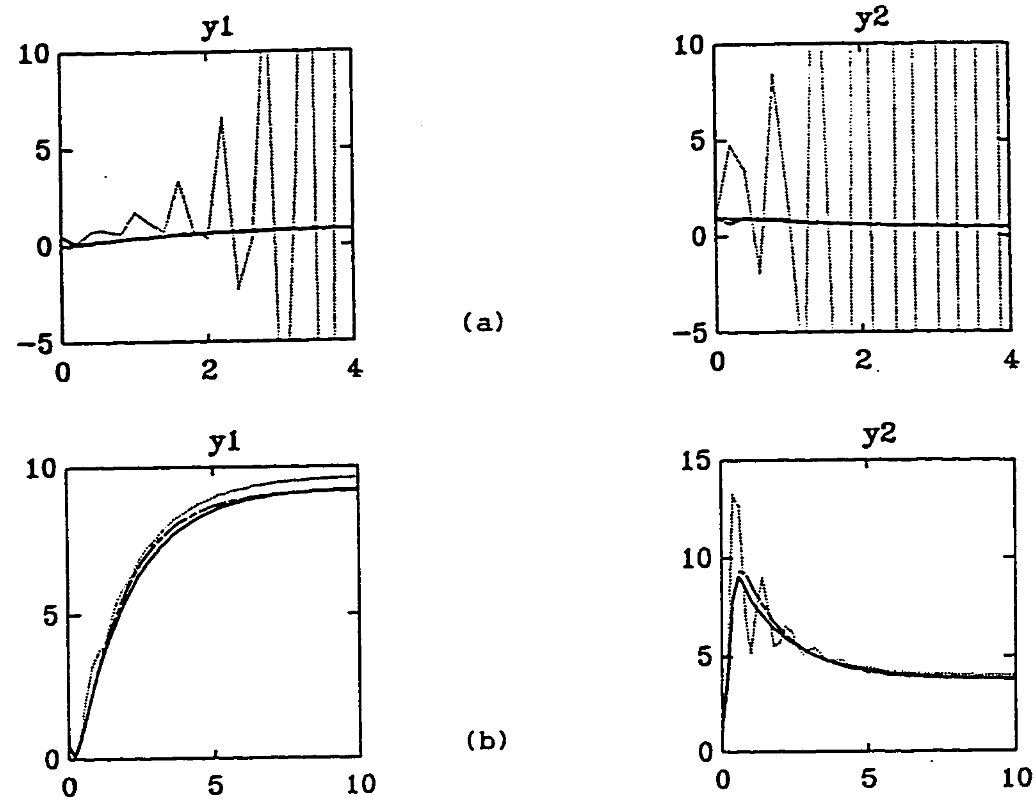


Fig.4.7. 1 Closed Loop Responses: Optimal (–), Suboptimal Balanced (–.), SP. (..), and 2–time Scale Bal. (––); u=0.1, (a) R = I. (b) R = 100I

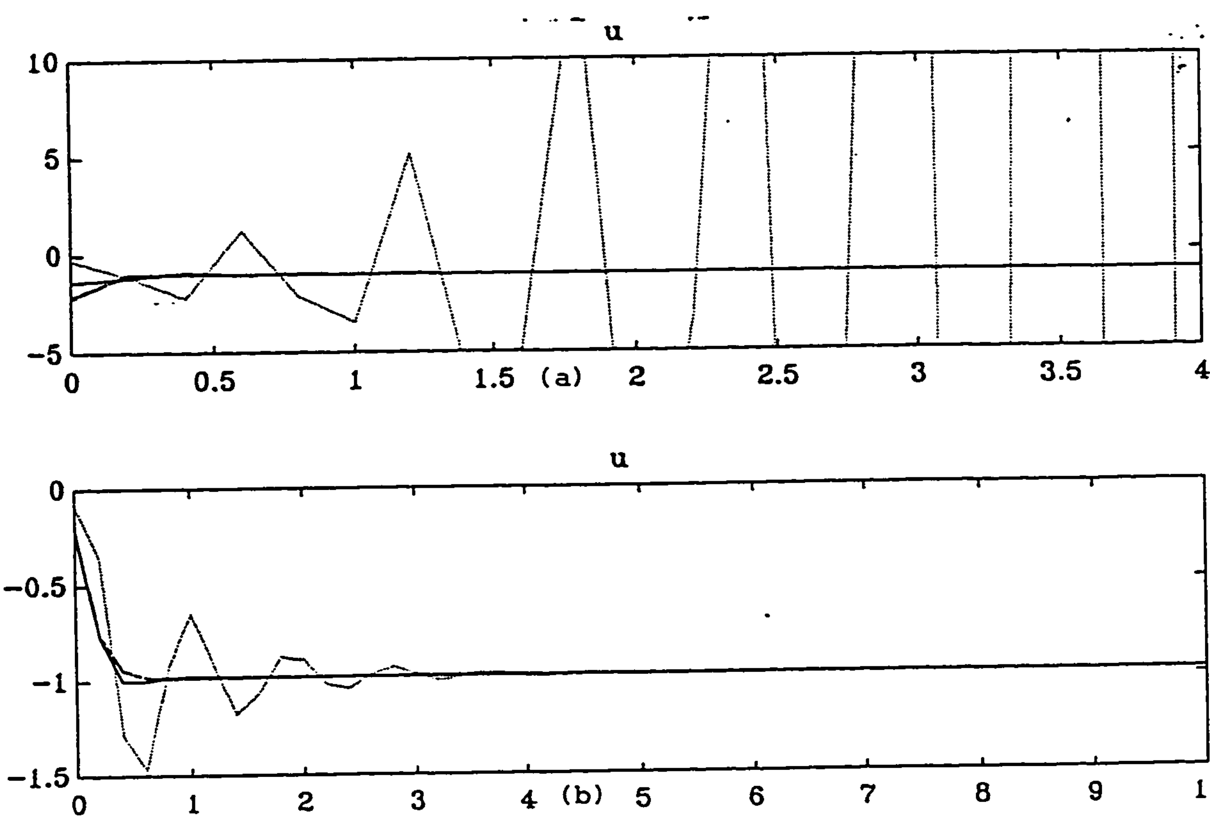
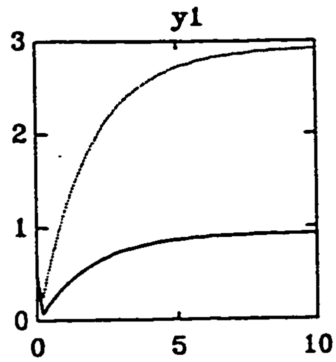


Fig.4 .7.2 Closed Loop Controls: Optimal (–), Suboptimal Balanced (–.), SP. (..), and 2–time Scale Bal (––); u=0.1. (a) R = I. (b) R =100I



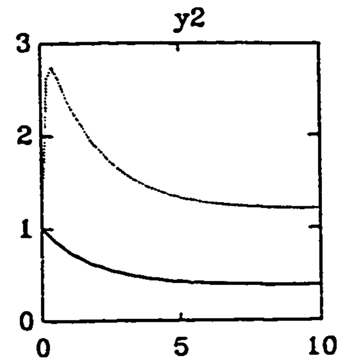


Fig. 4.7.3 Closed Loop Responses: Optimal (–), Suboptimal Balanced (–.), SP. (..), and 2–time Scale Bal. (––); μ=.02 , R = I

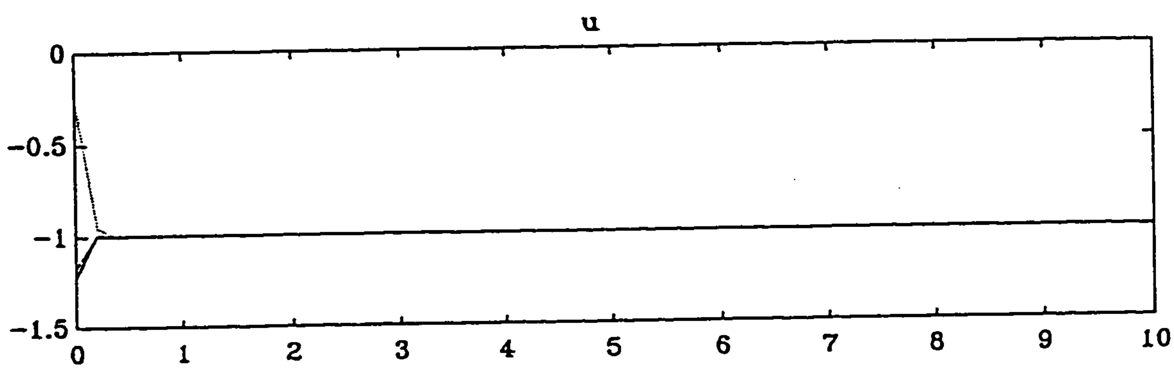


Fig. 4.7.4 Closed Loop Controls: Optimal (–), Suboptimal Balanced (–.), SP. (..), and 2–time Scale Bal (––); μ=.02 , R = I

4.8 Comparison of Discrete Suboptimal Control Laws Based on Balancing and Singular Perturbation: Simulation and Results

In the following, we are going to simulate a fifth order discrete steam power plant which is in the singularly perturbed form (3.3.21). For balancing, two types of reduced order models will be considered. The first one is the one developed in [26] which we are going to refer to as balancing. It is the discrete version of (4.2.5). This reduced order model is known to be unbalanced (theorem 2.5). The other type is the 2-time scale approximation of the balanced system as given in (2.4.25) with $\alpha = -1$ [36-37]. This reduced order model in known to be balanced (theorem 2.6). The discrete version of the suboptimal control law (4.2.15) will be applied to both model. For singular perturbation, the suboptimal control law (3.3.29) will be applied to the slow subsystem of the unbalanced singularly perturbed system given in (3.3.22)

4.8.1 Steam Power Plant

The following example is the singular perturbation form of the system given in [23]. The system matrices in singularly perturbed form are given as

$$A_{11} = \begin{bmatrix} 0.915 & 0.051 \\ -0.03 & 0.889 \end{bmatrix} \qquad A_{12} = \begin{bmatrix} 0.038 & 0.015 & 0.038 \\ -0.0005 & 0.046 & 0.111 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} -0.0016 & 0.1238 \\ -0.1892 & -0.0058 \\ -0.0392 & -0.0008 \end{bmatrix} \quad A_{22} = \begin{bmatrix} 0.0654 & 0.0037 & 0.0127 \\ -0.0056 & 0.0635 & -0.0064 \\ -0.0011 & 0.0238 & 0.0069 \end{bmatrix} \tag{4.8.1a}$$

$$B_1' = [0.0098 \quad 0.122] \;,\; B_2' = [0.0095 \quad 0.1487 \quad 0.0304] \tag{4.8.1b}$$

$$C_1 = [1 \quad 0 \quad 0 \quad 0 \quad 1] \;,\; C_2 = [0 \quad 1 \quad 0 \quad 1 \quad 0] \tag{4.8.1c}$$

Also, the initial condition vector x_0 is

$$x_0' = [0.5 \quad 0.5 \quad 0.5 \quad 0.5 \quad 0.5] \tag{4.8.1d}$$

The open loop eigenvalues *eva* are:

eva = [8.9295e-1±9.4319e-2i 2.9789e-2 2.5066e-1±2.5515e-2i]

and the singular values of the balanced system Σ at $\mu = 0.2646$ are

$$\Sigma_{\mu=0.2646} = [9.5739e-1 \quad 5.5193e-1 \quad 1.7160e-1 \quad 2.0809e-2 \quad 5.0833e-5]$$

From the open loop eigenvalues, it is possible to see that the system can be modeled by two subsystems: slow subsystem of order $r=2$ and fast subsystem of order $r=3$. Two cases will be considered : $R = I_m$, and $R = 0.01 * I_m$ both with $\mu = 0.2646$. In every case the full order system will be reduced to a second order system using balancing, 2-time scale balancing [36-37], and using singular perturbation. Then, the suboptimal control laws obtained using the different reduced order models will be applied to the full order system and the performance characteristics will be evaluated.

4.8.1.1 Simulation and Result

Table (4.8.1), gives a summary of the different characteristics of the optimal and the suboptimal closed loop systems using balancing, 2-time scale balancing, and singular perturbation. Comparing the characteristics of the suboptimal closed loop system obtained by balancing, 2-time scale balancing, and singular perturbation to those of the optimal closed loop system, we see the following:

1. Stability: All the methods give stable suboptimal closed loop systems.

2. Performance degradation dJ: It is small for balancing and 2-time scale balancing while it is very big for singular perturbation.

3. Suboptimality degree ϵ: For $R = I_m$, it is good for balancing and 2-time scale balancing while it is very small for singular perturbation. For $R = 0.01 * I_m$, it is small for balancing and 2-time scale balancing while it is extremely small for singular perturbation.

4. Closed loop eigenvalues: It can be seen that all the methods resulted in eigenvalues far from optimum.

5. Suboptimal step responses and controls: Suboptimal responses based on balancing and 2-time scale balancing are extremely close to the optimal ones when $R = I_m$ and $R = 0.01 * I_m$ but the control has shifted very little from optimality for balancing. However, suboptimal responses and controls based on singular perturbation are far from optimum for both cases.

Suboptimal Balanced Characteristics	Suboptimal 2-time Scale Balanced Characteristics	Suboptimal Singular Perturbation Characteristics
	$R = I_m$	
$\mathcal{J}$=4.1525e+0 dJ=8.8507e-1% ϵ=7.8927e-1	$\mathcal{J}$=4.2584e+0 dJ=3.4594e+0% ϵ=7.7144e-1	$\mathcal{J}$=1.3368e+1 dJ=2.2479e+2% ϵ=2.1625e-2
	Eigenvalues	
8.6249e-1±5.123e-2 i 3.0014e-2 1.2816e-1 2.4270e-1	9.0345e-1 8.0744e-1 1.7102e-2 7.6306e-2 2.4480e-1	9.6594e-1 8.4282e-1 3.4516e-2 2.2683e-1 2.6003e-1
	$R = 0.01 * I_m$	
$\mathcal{J}$=3.7617e+0 dJ=1.1678e+0% ϵ=3.0898e-1	$\mathcal{J}$=4.1050e+0 dJ=1.0403e+1% ϵ=2.7115e-1	$\mathcal{J}$=1.1697e+1 dJ=2.1458e+2% ϵ=4.7144e-3
	Eigenvalues	
9.0734e-1 7.5610e-1 -1.5948e-1 2.8605e-2 2.4672e-1	9.3746e-1 -4.2452e-1 7.0738e-1 3.2095e-2 2.4755e-1	9.7592e-1 8.3486e-1 3.4989e-2 2.2354e-1 2.6198e-1

Optimal Characteristics:

$R = I_m$: $J^* = 4.1160e+0$

Eigenvalues = [8.6990e-1 8.3068e-1 2.9857e-2 1.8826e-1 2.4012e-1]

$R = 0.01 * I_m$: $J^* = 3.7182e+0$

Eigenvalues = [9.1786e-1 7.1053e-1 6.6460e-3 2.8252e-2 2.4795e-1]

TABLE 4.8.1: Steam Power Plant Suboptimal Characteristics Based on Balancing, 2-time Scale Balancing, and Singular Perturbation for μ= 0.2646

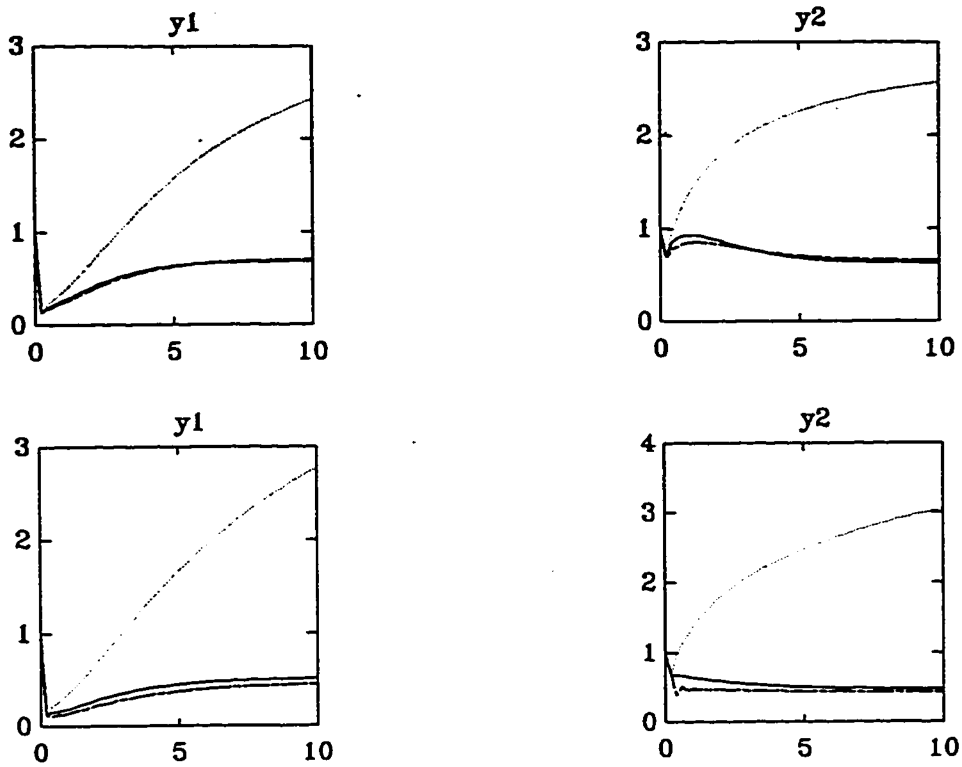


Fig.4.8.1 Closed Loop Responses: Optimal (–), Suboptimal Balanced (–.), SP. (..), and 2–time Scale Bal. (––); u=0.2646; (a) R=I, (b) R=0.01I

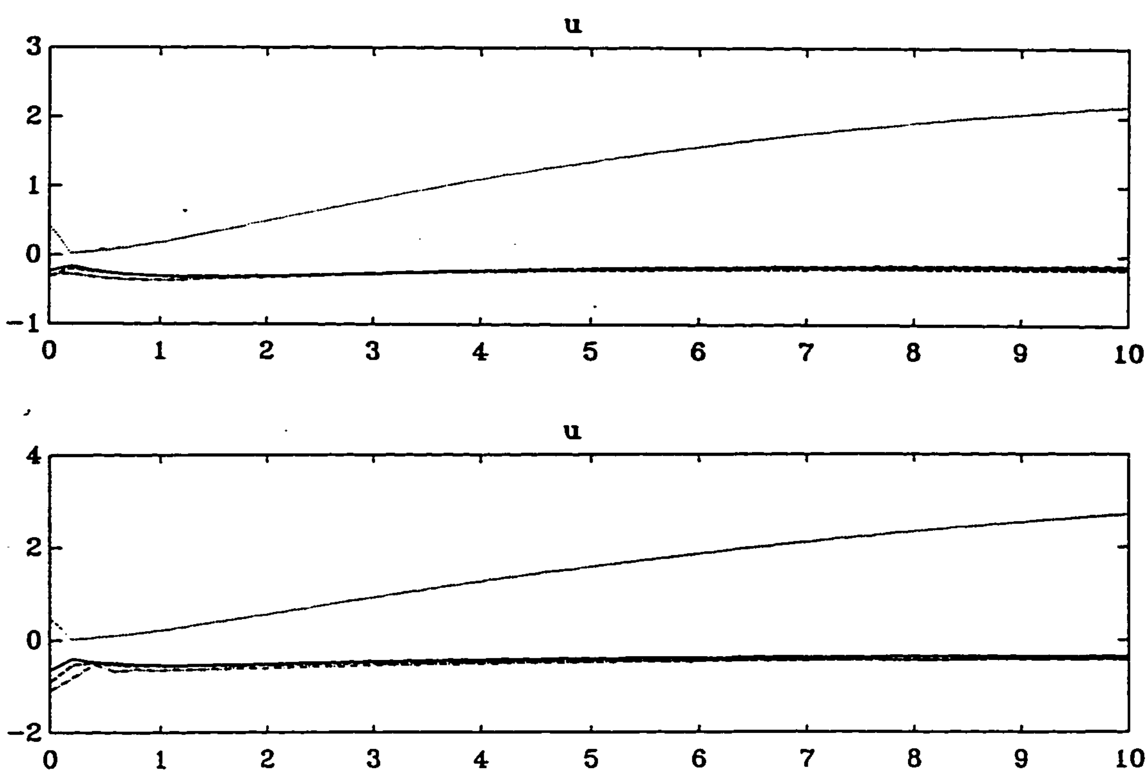


Fig.4.8.2 Closed Loop Controls: Optimal (–), Suboptimal Balanced (–.), SP. (..), and 2–time Scale Bal (––); u=0.2646; (a) R=I, (b) R=0.01I

4.9 Summary

Suboptimal control law based on balancing has been derived for the first time. The performance of the suboptimal closed loop system based on balancing has been compared to those obtained by aggregation and singular perturbation. It has been seen that the performance of the newly developed method had performed much better than the other methods. In fact, the suboptimal responses and controls of the suboptimal closed loop system based on balancing had been extremely close to the optimal ones.

Although, extensive simulation has shown that the resulting suboptimal closed loop system based on balancing had been stable, stability has not been proven theoretically. To overcome this problem, we have introduced a new reduction method called Balagg which combines features of balancing and aggregation. Suboptimal closed loop system based on this method is proven to be stable and the responses were better than aggregation and sometimes even better than balancing.

Numerous examples, most of which are physical plants, have been simulated. The results that were obtained have been very consistent. Only, five plants have been presented in this chapter due to the limited size of the thesis.

CHAPTER 5

Pole Placement Based on Reduced Order Models

5.1 Introduction

Pole placement is a method for shaping the open loop responses. If a system is controllable and all the states are available for measurement (else an observer is used) then it is possible to have a control law of the form $u=-K_{po}x$ which can place the open loop poles to any desired locations. In principle, it is possible to stabilize an unstable system, to increase the speed of the response, or to add damping to a system having poles near the imaginary axis. In practice, the desired pole locations might not be feasible. For example, to increase the speed of a very slow system, then a large control signal will be required which the input source might not be able to deliver.

A major problem with pole placement is that it is not robust. It is known that the gain K_{po} is very sensitive to the location of the open loop poles. Therefore, slight changes in the open loop pole locations may cause the closed loop behavior to be completely different from that it was designed for [78].

There are different algorithms to do pole placement [73][78-80]. In the following we are going to use Kautsky and Nichols algorithm [73] due to its numerical stability.

Pole Placement based on the full order system usually gives rise to complicated controllers. Therefore, pole placement based on reduced order models can be used in order to simplify the controller.

In this chapter, we are going to look at two different areas in which pole placement based on reduced order models can be used. The first situation is when it is desired to assign only r eigenvalues of the n open loop eigenvalues $(r < n)$. The second case is when it is desired to shape the open loop response to a nominal response like the optimal response. In this case, instead of assigning all the open loop eigenvalues to the nominal locations, only the eigenvalues of the reduced order models are assigned. A control law is then developed from the reduced order controller and then applied to the full order open loop system. The response of the resulting closed loop system is then compared to the nominal one.

In chapter 3, the theories of pole placement based on reduced order models obtained by aggregation and singular perturbation were presented. In this chapter, pole placement based on the reduced order models obtained by aggregation, balancing, balagg, and singular perturbation will be simulated and evaluated. The effect of applying the suboptimal controllers based on the reduced order models to the full order system will also be studied. Section 5.2 will give extensive simulation results of pole placement based on reduced order models obtained by balancing. In section 5.3, a comparison of suboptimal responses based on reduced order models obtained by balancing, aggregation and balagg to nominal responses will be given. In section 5.4, the comparison will be done for suboptimal responses based on reduced order models obtained by balancing, and singular perturbation.

5.2 Pole Placement Based on Balancing

Pole placement based on reduced order models obtained by balancing has not been considered in the literature. In this section, we are going to present simulation results of pole placement based on balancing.

Consider the controllable, observable, and stable system Sl given in (4.2.1). This system can be transformed into a balanced system Sb given in (4.2.2) with controllability and observability grammian Σ. A reduced order model of Sb can be obtained as given in (4.2.5). Let the open loop eigenvalues of the balanced system be $eva=[\lambda_1,\ldots,\lambda_n]$ and the reduced order model eigenvalues be $evar=[\bar{\lambda}_1,\ldots,\bar{\lambda}_r]$. It is desired to place r eigenvalues of eva to $[\lambda_{s1},\ldots,\lambda_{sr}]$ but using the reduced order model. The eigenvalues of the resulting closed loop system should be $[\lambda_{s1},\ldots,\lambda_{sr}]$ and the untouched $(n-r)$ eigenvalues of eva.

The followings are the control laws which will be used to achieve the design purpose:

The control law that will be applied to the reduced order model (4.2.5) to assign its poles $evar=[\bar{\lambda}_1,\ldots,\bar{\lambda}_r]$ to the desired locations $[\lambda_{s1},\ldots,\lambda_{sr}]$ is:

$$u = -k_{bp}x_1 \tag{5.2.1}$$

The control law that will be applied to the full order system is:

$$u = -k_{bps}x \tag{5.2.2}$$

where $$k_{bps} = [k_{bp}|0]$$

The eigenvalues of the resulting closed loop system should be $[\lambda_{s1},\ldots,\lambda_{sr}]$ and the untouched $(n-r)$ eigenvalues of eva .

5.2.1 Continuous Time Case: Simulation and Results

The control laws (5.2.1) and (5.2.2) have been applied to several continuous examples. The simulation of the control laws is done using the MatLab package.

5.2.1.1 Simulation

Example 5.2.1

Consider the balanced system given below

$$A_b = \begin{bmatrix} -9.8146e-001 & -1.7893e+000 & -1.6662e+000 & -8.0401e-002 \\ -1.7893e+000 & -4.9064e+000 & -5.6040e+000 & -3.4811e-001 \\ -1.6662e+000 & -5.6040e+000 & -7.5580e+000 & -6.5319e-001 \\ -8.0401e-002 & -3.4811e-001 & -6.5319e-001 & -3.8548e+002 \end{bmatrix}$$

$B_b' =$ [3.4318e+002 3.9624e+002 3.2529e+002 1.4057e+001]

$C_b =$ [3.4318e+002 3.9624e+002 3.2529e+002 1.4057e+001]

$\Sigma =$ [6.0000e+004 1.6000e+004 7.0000e+003 2.5630e-001]

$eva =$ [-1.1194e-001 -8.3534e-001 -1.2497e+001 -3.8548e+002]

$evar =$ [-1.1194e-001 -8.3540e-001 -1.2499e+001]

It can be seen that the eigenvalues of the reduced order model *evar* are the dominant eigenvalues of the open loop system eigenvalues *eva*. The results of the simulation will be tabulated in table 5.2.1.

<u>Example 5.2.2</u>

Consider the balanced system given below

$$A_b = \begin{bmatrix} -4.2153e+001 & -2.6683e+001 & -2.4848e+001 & -6.4321e-003 \\ -2.6683e+001 & -2.5406e+001 & -2.9018e+001 & -9.6698e-003 \\ -2.4848e+001 & -2.9018e+001 & -3.9137e+001 & -1.8144e-002 \\ -6.4321e-003 & -9.6698e-003 & -1.8144e-002 & -5.7440e-002 \end{bmatrix}$$

$B_b' =$ [2.2491e+003 9.0167e+002 7.4021e+002 1.7159e-001]

$C_b =$ [2.2491e+003 9.0167e+002 7.4021e+002 1.7159e-001]

$\Sigma =$ [6.0000e+004 1.6000e+004 7.0000e+003 2.5630e-001]

$eva =$ [-9.3153e-001 -8.9646e+001 -1.6119e+001 -5.7427e-002]

$evar =$ [-9.3152e-001 -8.9646e+001 -1.6119e+001]

It can be seen that the eigenvalues of the reduced order model *evar* are the non dominant eigenvalues of the open loop system eigenvalues *eva*. The results of the simulation will be tabulated in table 5.2.2.

Example 5.2.3

Consider the balanced system given below

$$A_b = \begin{bmatrix} -1.1680e+001 & -1.1716e+001 & -1.3290e+001 & -2.6032e-002 \\ -1.1716e+001 & -1.7678e+001 & -2.4596e+001 & -6.2017e-002 \\ -1.3290e+001 & -2.4596e+001 & -4.0407e+001 & -1.4175e-001 \\ -2.6032e-002 & -6.2017e-002 & -1.4175e-001 & -3.3955e+000 \end{bmatrix}$$

$B_b' =$ [1.1839e+003 7.5213e+002 7.5213e+002 1.3193e+000]

$C_b =$ [1.1839e+003 7.5213e+002 7.5213e+002 1.3193e+000]

$\Sigma =$ [6.0000e+004 1.6000e+004 7.0000e+003 2.5630e-001]

eva = [-6.2176e-001 -6.9264e+000 -6.2218e+001 -3.3949e+000]

evar = [-6.2181e-001 -6.9260e+000 -6.2218e+001]

It can be seen that the eigenvalues of the reduced order model *evar* are a mixture of the open loop system eigenvalues *eva* i.e it contains dominant and non dominant open loop eigenvalues. The results of the simulation will be tabulated in table 5.2.3.

In examples (5.2.1-5.2.3), it is desired to reassign the first three open loop eigenvalues to new locations. The eigenvalues of the resulting closed loop system should be the desired ones and the fourth untouched open loop eigenvalue. The desired poles will be

$$[\lambda_{s1},\ldots,\lambda_{s3}] \;=\; N*[\lambda_1,\ldots,\lambda_3] \qquad (5.2.3)$$

where $N = 10e{+}i$; i *is an integer with* $-4 \leq i \leq 4$ & $i \neq 0$

To achieve the design purpose, a reduced order model of order three is first obtained as in (4.2.5). Then the poles of the reduced order models are relocated to the desired locations given in (5.2.3). The control law that achieves the placement is as given in (5.2.1). After that, a control law is obtained from (5.2.1) as given in (5.2.2) and then applied to the full order system (A_b, B_b, C_b). The resulting closed loop system ($A_b - B_b*K_{bps}$) matrix should have $[\lambda_{s1},\ldots,\lambda_{s3}]$ and λ_4 as its eigenvalues. The tables and the results of the simulation are given next.

5.2.1.2 Results

Tables (5.2.1)-(5.2.3) give a summary of the desired closed loop eigenvalues, obtained closed loop eigenvalues, and the gains obtained from (5.2.2) for examples (5.2.1)-(5.2.3) respectively. The following observations can be made.

1. In example 5.2.1, when N=10,100,0.1 the resulting closed loop eigenvalues are almost identical to the desired eigenvalues. As the poles are assigned very far from their original positions (N=1000,10000), the deign purpose is not achieved and, in fact, the system becomes unstable for N=10000. In these two case, the feedback gain becomes very big for N=1000 and extremely big for N=10000. For N=0.01,0.001,0.0001,

the system becomes unstable. Also, the feedback gain becomes very small and it seems that it reaches saturation and becomes almost equal for all the three cases.

2. In example 5.2.2, when N=10,100,1000,10000,0.1 the resulting closed loop eigenvalues are almost identical to the desired eigenvalues. Also, the feedback gain is extremely high for some of the cases. For N=0.01,0.001,0.0001, the system becomes unstable. Also, the feedback gain becomes small and saturated.

3. In example 5.2.3, when N=10,100,1000,10000 the resulting closed loop eigenvalues are almost identical to the desired eigenvalues. Also, the feedback gain is extremely high for some of the cases. For N=0.1, they are not accurately placed but the system is stable. For N=0.01,0.001,0.0001, the system becomes unstable. Also, the feedback gain becomes small and saturated.

Desired Eigenvalues	Closed Loop Eigenvalues
[10e+1*(-1.1194e-1 -8.3534e-1 -1.2497e+1) -3.8548e+2]	[-1.1196e+0 -8.3533e+0 -1.2496e+2 -3.8549e+2]
[10e+2*(-1.1194e-1 -8.3534e-1 -1.2497e+1) -3.8548e+2]	[-1.1175e+1 -8.4186e+1 -1.2523e+3 -3.8230e+2]
[10e+3*(-1.1194e-1 -8.3534e-1 -1.2497e+1) -3.8548e+2]	[-1.2696e+4 -5.1618e+2±2.8784e+2i -1.0156e+2]
[10e+4*(-1.1194e-1 -8.3534e-1 -1.2497e+1) -3.8548e+2]	U
[10e-1*(-1.1194e-1 -8.3534e-1 -1.2497e+1) -3.8548e+2]	[-1.0551e-2 -8.4726e-2 -1.2483e+0 -3.8548e+2]
[10e-2*(-1.1194e-1 -8.3534e-1 -1.2497e+1) -3.8548e+2]	U
[10e-3*(-1.1194e-1 -8.3534e-1 -1.2497e+1) -3.8548e+2]	U
[10e-4*(-1.1194e-1 -8.3534e-1 -1.2497e+1) -3.8548e+2]	U

Feedback Gains in The Same Sequence as the Table Entries:

1. K_{pbs} = [8.1332e-1 -8.4697e-1 5.4561e-1 0]
2. K_{pbs} = [6.2573e+2 -1.1662e+3 7.6448e+2 0]
3. K_{pbs} = [6.0851e+5 -1.1875e+6 8.0425e+5 0]
4. K_{pbs} = [6.0681e+8 -1.1895e+9 8.0872e+8 0]
5. K_{pbs} = [-4.5047e-3 -1.3221e-2 -1.6345e-2 0]
6. K_{pbs} = [-4.7267e-3 -1.4463e-2 -1.8318e-2 0]
7. K_{pbs} = [-4.7472e-3 -1.4585e-2 -1.8519e-2 0]
8. K_{pbs} = [-4.7492e-3 -1.4598e-2 -1.8539e-2 0]

Table 5.2.1: Desired Eigenvalues, Closed Loop Eigenvalues, and Feedback Gains for example 5.2.1

134
0093

Desired Eigenvalues	Closed Loop Eigenvalues
[10e+1(-9.3153e-1 -8.9646e+1 -1.6119e+1) -5.7427e-2]	[-8.9646e+2 -1.6119e+2 -9.3153e+0 -5.7429e-2]
[10e+2(-9.3153e-1 -8.9646e+1 -1.6119e+1) -5.7427e-2]	[-8.9646e+3 -1.6119e+3 -9.3153e+1 -5.7429e-2]
[10e+3(-9.3153e-1 -8.9646e+1 -1.6119e+1) -5.7427e-2]	[-8.9646e+4 -1.6119e+4 -9.3153e+2 -5.7429e-2]
[10e+4(-9.3153e-1 -8.9646e+1 -1.6119e+1) -5.7427e-2]	[-8.9900e+5 -1.5852e+5 -9.4447e+3 -5.7429e-2]
[10e-1(-9.3153e-1 -8.9646e+1 -1.6119e+1) -5.7427e-2]	[-8.9646e+0 -1.6115e+0 -9.7976e-2 -5.2950e-2]
[10e-2(-9.3153e-1 -8.9646e+1 -1.6119e+1) -5.7427e-2]	U
[10e-3(-9.3153e-1 -8.9646e+1 -1.6119e+1) -5.7427e-2]	U
[10e-4(-9.3153e-1 -8.9646e+1 -1.6119e+1) -5.7427e-2]	U

Feedback Gains in The Same Sequence as the Table Entries:

1. K_{pbs} = [1.6616e+0 -4.4773e+0 1.7025e+0 0]
2. K_{pbs} = [7.8184e+2 -4.5576e+3 3.1904e+3 0]
3. K_{pbs} = [7.0729e+5 -4.5390e+6 3.3802e+6 0]
4. K_{pbs} = [6.9999e+8 -4.5369e+9 3.3996e+9 0]
5. K_{pbs} = [-2.4945e-2 -2.2514e-2 -2.6511e-2 0]
6. K_{pbs} = [-2.6649e-2 -2.5430e-2 -3.0753e-2 0]
7. K_{pbs} = [-2.6813e-2 -2.5724e-2 -3.1196e-2 0]
8. K_{pbs} = [-2.6829e-2 -2.5753e-2 -3.1240e-2 0]

Table 5.2.2: Desired Eigenvalues, Closed Loop Eigenvalues, and Feedback Gains for example 5.2.2

0093

Desired Eigenvalues	Closed Loop Eigenvalues
[10e+1*(-6.2176e-1 -6.9264e+0 -6.2218e+1) -3.3949e+0]	[-6.2218e+2-6.9264e+1 -6.2185e+0 -3.3943e+0]
[10e+2*(-6.2176e-1 -6.9264e+0 -6.2218e+1) -3.3949e+0]	[-6.2218e+3 -6.9262e+2 -6.2180e+1 -3.3947e+0]
[10e+3*(-6.2176e-1 -6.9264e+0 -6.2218e+1) -3.3949e+0]	[-6.2220e+4 -6.9240e+3 -6.2197e+2 -3.3948e+0]
[10e+4*(-6.2176e-1 -6.9264e+0 -6.2218e+1) -3.3949e+0]	[-6.2473e+5 -6.6477e+4 -6.4515e+3 -3.3948e+0]
[10e-1*(-6.2176e-1 -6.9264e+0 -6.2218e+1) -3.3949e+0]	[-6.2327e+0 -5.8789e-2 -7.0265e-1 -3.3780e+0]
[10e-2*(-6.2176e-1 -6.9264e+0 -6.2218e+1) -3.3949e+0]	U
[10e-3*(-6.2176e-1 -6.9264e+0 -6.2218e+1) -3.3949e+0]	U
[10e-4*(-6.2176e-1 -6.9264e+0 -6.2218e+1) -3.3949e+0]	U

Feedback Gains in The Same Sequence as the Table Entries:

1. K_{pbs} = [1.6454e+0 -2.9860e+0 1.2309e+0 0]
2. K_{pbs} = [1.0317e+3 -3.5038e+3 1.8890e+3 0]
3. K_{pbs} = [9.7670e+5 -3.5342e+6 1.9969e+6 0]
4. K_{pbs} = [9.7128e+8 -3.5369e+9 2.0081e+9 0]
5. K_{pbs} = [-1.4931e-2 -2.3846e-2 -3.6132e-2 0]
6. K_{pbs} = [-1.5792e-2 -2.6307e-2 -4.0664e-2 0]
7. K_{pbs} = [-1.5873e-2 -2.6551e-2 -4.1129e-2 0]
8. K_{pbs} = [-1.5881e-2 -2.6575e-2 -4.1175e-2 0]

Table 5.2.3: Desired Eigenvalues, Closed Loop Eigenvalues, and Feedback Gains for example 5.2.3

There is no answer to when the resulting closed loop system becomes unstable. It seems that the stability depends on the separation of the singular values i.e. σ_3 and σ_4 and on the relative stability of every open loop eigenvalue. When the relative stability is very small and if the placement is toward decreasing the relative stability (i.e. N=0.01,0.001,0.0001) the systems become unstable. To see the effect of the singular value separation i.e. σ_3 and σ_4 on the stability, the simulation was done on other three examples having the same eigenvalues as examples (5.2.1-5.2.3) but the separation between σ_3 and σ_4 has been increased. The new singular values were made to be

$$\Sigma = [6.0000e+004 \quad 1.6000e+4 \quad 7.0000e+3 \quad 2.5630e-3]$$

The results of the simulation (which are not shown here due to the lack of space) show that more stable closed loop systems are obtained.

5.2.2 Discrete Time Case: Simulation and Results

The control laws (5.2.1) and (5.2.2) have been applied to several discrete examples. The simulation of the control laws is done using the MatLab package.

5.2.2.1 Simulation

Example 5.2.4

Consider the discrete time balanced system given as

$$A_b = \begin{bmatrix} 9.9008e-001 & -1.8058e-002 & 1.5613e-002 & -3.4544e-004 \\ -1.8058e-002 & 9.5033e-001 & 5.3077e-002 & -1.5498e-003 \\ 1.5613e-002 & 5.3077e-002 & 9.3266e-001 & 2.7831e-003 \\ -3.4544e-004 & -1.5498e-003 & 2.7831e-003 & 2.1190e-002 \end{bmatrix}$$

B_b' = [3.4396e+001 3.9352e+001 -2.9711e+001 5.7707e-001]

C_b = [3.4396e+001 3.9352e+001 -2.9711e+001 5.7707e-001]

Σ = [6.0299e+004 1.6401e+004 7.2504e+003 4.3595e-001]

eva = [8.8252e-001 9.9888e-001 9.9168e-001 2.1178e-002]

$evar$ = [8.8251e-001 9.9888e-001 9.9168e-001]

It can be seen that the eigenvalues of the reduced order model $evar$ are the dominant eigenvalues of the open loop system eigenvalues eva.

The results of the simulation will be tabulated in table 5.2.4.

Example 5.2.5

Consider the discrete time balanced system given as

$$A_b = \begin{bmatrix} 2.8196e-001 & -2.1339e-001 & 1.7272e-001 & -3.0575e-005 \\ -2.1339e-001 & 8.0310e-001 & 2.6786e-001 & -9.7068e-005 \\ 1.7272e-001 & 2.6786e-001 & 4.3086e-001 & 4.1500e-004 \\ -3.0575e-005 & -9.7068e-005 & 4.1500e-004 & 9.9771e-001 \end{bmatrix}$$

B_b' = [3.0286e+002 2.8370e+001 -1.4431e+001 4.8026e-004]

C_b = [3.0286e+002 2.8370e+001 -1.4431e+001 4.8026e-004]

Σ = [1.0062e+005 1.6259e+004 5.3742e+003 2.5587e-001]

eva = [2.7714e-002 5.2478e-001 9.6342e-001 9.9771e-001]

$evar$ = [2.7714e-002 5.2478e-001 9.6342e-001]

It can be seen that the eigenvalues of the reduced order model $evar$ are relatively the non dominant eigenvalues of the open loop system eigenvalues eva.

The results of the simulation will be given in table 5.2.5.

Example 5.2.6

Consider the discrete time balanced system given as

$$A_b = \begin{bmatrix} 4.0951e-001 & -2.3511e-001 & 1.5088e-001 & -1.3590e-004 \\ -2.3511e-001 & 7.6747e-001 & 2.7415e-001 & -5.7456e-004 \\ 1.5088e-001 & 2.7415e-001 & 2.6502e-001 & 4.4117e-003 \\ -1.3590e-004 & -5.7456e-004 & 4.4117e-003 & 7.1209e-001 \end{bmatrix}$$

B_b' = [2.7484e+002 4.3235e+001 -1.7132e+001 2.7770e-004]

C_b = [2.7484e+002 4.3235e+001 -1.7132e+001 2.7770e-004]

Σ = [9.2041e+004 1.7659e+004 3.9968e+003 1.7309e-001]

eva = [1.9856e-003 9.3972e-001 5.0025e-001 7.1214e-001]

$evar$ = [5.0028e-001 2.0037e-003 9.3971e-001]

It can be seen that the eigenvalues of the reduced order model $evar$ are a mixture of different eigenvalues.

The results of the simulation will be tabulated in table 5.2.6.

5.2.2.2.Results

Tables 5.2.4-5.2.6 give a summary of the desired closed loop eigenvalues, obtained closed loop eigenvalues, and the gains obtained from (5.2.2) for examples 5.2.4-5.2.6 respectively. The following observations can be made.

1. In example 5.2.4, the pole placement resulted in a stable closed loop system but the poles are not placed at the desired locations.

2. In example 5.2.5, in all of the cases, the designed purpose has been achieved. The feedback gain is small for all of them.

3. In example 5.2.6, in the first case, the designed purpose has been achieved except for the first desired eigenvalue. In the other two cases, the resulting closed loop eigenvalues are extremely close to the desired ones.

In the next section, a comparison of suboptimal responses based on reduced order models obtained by balancing, aggregation and balagg to nominal responses will be given.

133

Desired Eigenvalues	Closed Loop Eigenvalues
[2.2063e-1 2.4972e-1 2.4792e-1 2.1178e-2]	[2.5266e-1±3.2171e-1i 4.7686e-1 -2.4271e-1]
[4.4126e-1 4.9944e-1 4.9584e-1 2.1178e-2]	[4.3975e-1±1.6573e-1i 6.0707e-1 -2.8827e-2]
[6.6189e-1 7.4916e-1 7.4376e-1 2.1178e-2]	[6.9121e-1±3.0920e-2i 7.7498e-1 1.8604e-2]

Feedback Gains for the Closed Loop Eigenvalues in the Same Sequence as the Table Entries:

1. k_{pbs} = [2.4439e+1 -4.6607e+1 -3.3511e+1 0]
2. k_{pbs} = [7.8804e+0 -1.4889e+1 -1.0646e+1 0]
3. k_{pbs} = [1.2338e+0 -2.2696e+0 -1.6019e+0 0]

Table 5.2.4: Desired Eigenvalues, Closed Loop Eigenvalues, and Feedback Gains for example 5.2.4

0093

Desired Eigenvalues	Closed Loop Eigenvalues
[4.9640e-4 2.3493e-1 1.2506e-1 7.1214e-1]	[9.7669e-5 2.3428e-1 1.2604e-1 7.1216e-1]
[9.9280e-4 4.6986e-1 2.5013e-1 7.1214e-1]	[9.3003e-4 4.6970e-1 2.5026e-1 7.1216e-1]
[1.4892e-3 7.0479e-1 3.7519e-1 7.1214e-1]	[1.4654e-3 7/0399e-1 3.7520e-1 7.1290e-1]

Feedback Gains for the Closed Loop Eigenvalues in the Same Sequence as the Table Entries:

1. k_{pbs} = [6.9644e-3 -2.2417e-2 -7.9740e-3 0]
2. k_{pbs} = [4.3450e-3 -1.2601e-2 -4.1823e-3 0]
3. k_{pbs} = [2.0237e-3 -5.1299e-3 -1.5254e-3 0]

Table 5.2.5: Desired Eigenvalues, Closed Loop Eigenvalues, and Feedback Gains for example 5.2.5

Desired Eigenvalues	Closed Loop Eigenvalues
[6.9285e-3 1.3120e-1 2.4086e-1 9.9771e-1]	[6.9235e-3 1.3121e-1 2.4085e-1 9.9771e-1]
[1.3857e-2 2.6239e-1 4.8171e-1 9.9771e-1]	[1.3856e-2 2.6239e-1 4.8171e-1 9.9771e-1]
[2.0785e-2 3.9359e-1 7.2257e-1 9.9771e-1]	[2.0785e-2 3.9359e-1 7.2257e-1 9.9771e-1]

Feedback Gains for the Closed Loop Eigenvalues in the Same Sequence as the Table Entries:

1. k_{pbs} = [6.2847e-3 -3.5488e-2 -1.6655e-2 0]
2. k_{pbs} = [3.9460e-3 -1.9896e-2 -8.8247e-3 0]
3. k_{pbs} = [1.8523e-3 -8.0909e-3 -3.2949e-3 0]

Table 5.2.6: Desired Eigenvalues, Closed Loop Eigenvalues, and Feedback Gains for example 5.2.6

5.3 Pole Placement Based on Reduced Order Models Obtained via Balancing, Aggregation, and Balagg: Comparison to Nominal Response

Pole placement based on reduced order models obtained by aggregation was discussed in chapter 3. It was stated that the closed loop eigenvalues are always the union of the desired eigenvalues and the untouched open loop ones (theorem 3.4).

Since in balagg, the reduced order model is obtained via aggregation reduction method, then also the closed loop eigenvalues will be the desired ones and the untouched open loop eigenvalues.

For balancing, it was seen in section (5.2) that the design purpose is not always achieved.

Now, the pole placement problem is going to be altered. One of the main reasons for basing the pole placement on reduced order models is to reduce the complexity of the controller. Suppose it is desired to shape the open loop response. One question that should be asked is what shape the open loop response should be designed to meet. A legitimate answer to the question is to shape the open loop response to the optimal response. In optimal control, all the open loop eigenvalues are usually placed. So to shape the open loop response to the optimal one, then all the poles should be placed to their optimal locations. Again, placing all the poles might result in a complicated controller. Therefore, to reduce the complexity of the controller, partial pole placement based on the reduced order model should be done. For this partial pole placement to be considered acceptable, it should result in a control law which when applied to the full order open loop system will result in responses close to the optimal ones.

In this section, the closeness of closed loop responses obtained from applying control laws derived from reduced order models obtained by aggregation, balancing and balagg will be compared to the optimal responses.

5.3.1 Continuous Time Case: Boiler

Consider the boiler plant studied in section 4.5.2. The open loop eigenvalues eva and the singular values of the balanced system Σ are:

eva = [-1.0262e+1±5.7148e+2i -5.0266e+2 -8.9874e+1 -1.5214e+1±1.1622e+1i -9.1000e-1 -4.4490e+0 -1.0987e+1]

Σ = [1.1924e+1 4.6040e+0 6.3901e-1 1.1925e-1 1.1563e-1 2.8093e-2 1.2317e-2 2.1317e-3 5.3668e-4].

Also the optimal eigenvalues for the case when $Q = I_9$ and $R = I_1$ are

[-1.2226e+1±5.7147e+2i -5.1129e+2 -2.3948e+2 -8.9705e+1 -1.9131e+1 -9.7065e-1 -4.4586e+0 -1.1096e+1]

It is desired to shape the open loop response to the optimal response. To avoid having a complex controller, the pole placement will be based on the reduced order models. Therefore, instead of placing all of the open loop eigenvalues, a reduced order model is first derived by balancing, aggregation, and balagg reduction methods. Then the eigenvalues of the reduced order models are assigned to the locations of the dominant optimal eigenvalues. Finally, a controller is derived from the reduced order controller and applied to the full order system.

For this example, a reduced order model of order $r=5$ is derived. The reduced order models obtained via aggregation and balagg will have the dominant open loop eigenvalues as its eigenvalues which are

$$evar = [-1.0262e+1 \pm 5.7148e+2i \quad -9.1000e-1 \quad -4.4490e+0 \quad -1.0987e+1]$$

The reduced order model obtained via balancing has the following eigenvalues

$$evar = [-9.1772e+0 \pm 5.7151e+2i \quad -9.2213e-1 \quad -1.6120e+1 \pm 1.1859e+1i]$$

The reduced order model eigenvalues will be assigned to

$$[-1.2226e+1 \pm 5.7147e+2i \quad -9.7065e-1 \quad -4.4586e+0 \quad -1.1096e+1]$$

A control law which assigns the reduced order model eigenvalues to the desired locations is first obtained. The control law that should be applied to the full order system is obtained from the reduced control law. For aggregation, and balagg, this is done by multiplying the reduced order control law by the aggregation matrix as discussed in chapters 3 and 4. For balancing, this is done by augmenting zeros to the reduced order controller as discussed earlier in this chapter and in chapter 4. The resulting closed loop eigenvalues based on aggregation and balagg are

$$evaf = [-1.2226e+1 \pm 5.7147e+2i \;\; -9.7065e-1 \;\; -4.4586e+0 \;\; -1.1096e+1$$
$$-5.0266e+2 \;\; -8.9874e+1 \;\; -1.5214e+1 \pm 1.1622e+1i]$$

While those based on balancing are

$$evaf = [-1.3493e+1 \pm 5.7147e+2i \;\; -5.0229e+2 \;\; -8.9933e+1 \;\; -1.2185e+1$$
$$-7.4396e+0 \pm 4.2410e+0i \;\; -1.5147e+0 \pm 5.0514e-1i \;]$$

Figures (5.3.1-5.3.2) show the full order closed loop system responses when the placement is done based on the reduced order models. Also, the optimal response based on the full order system is also shown.

It can be seen that non of the closed loop responses (i.e when the placement is based on the reduced order models) is close to the desired optimal responses. Therefore, if all of the open loop eigenvalues are to be placed, then pole placement based on reduced order models might not be a good approach.

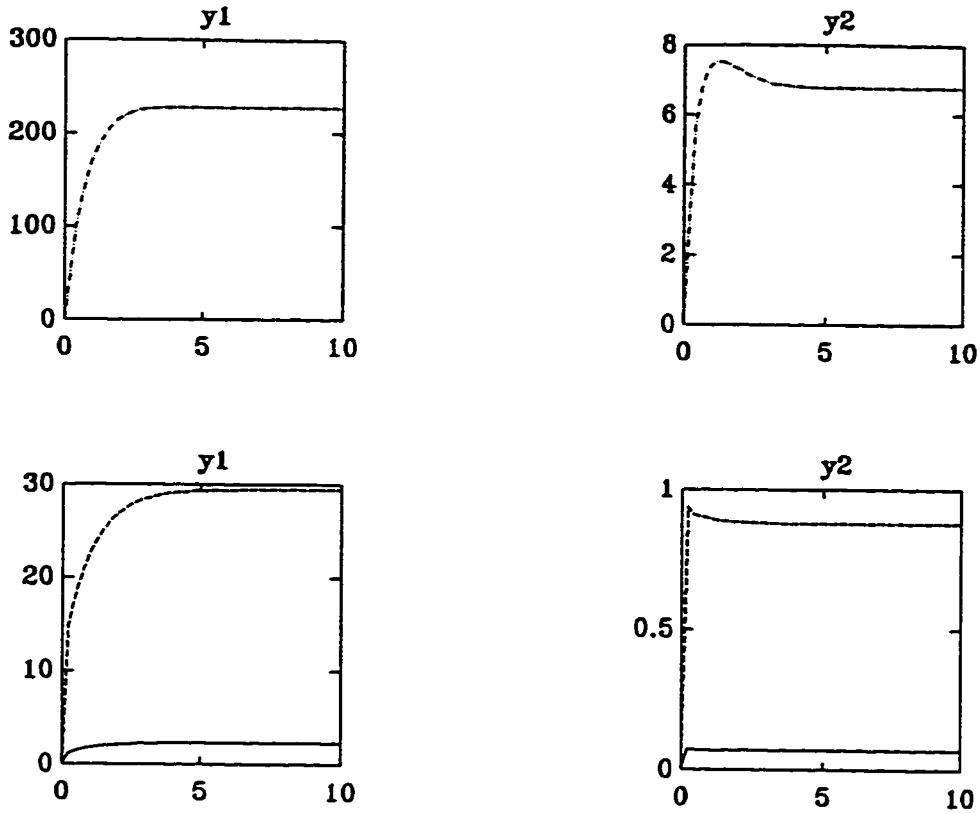


Fig.5.3.1 Closed Loop Responses. Pole Placement Based on: (a) Balancing (-.); (b) Full Order System (-), Aggregation (:), and Balagg (--) Based Systems; r = 4

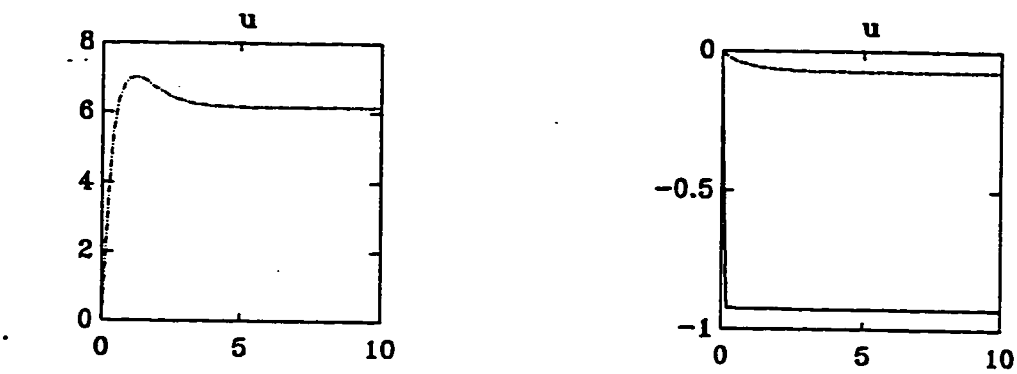


Fig.5.3.2 Closed Loop Controls. Pole Placement Based on: (a) Balancing (-.); (b) Full Order System (-), Aggregation (:), and Balagg (--) Based Systems; r = 4

5.3.2 Discrete Time Case: Chemical Reactor

Consider the discrete chemical reactor studied in section 4.6.1. The open loop eigenvalues *eva* and the singular values of the balanced system Σ are:

eva = [6.7322e-1±1.7791e-1i 5.1456e-1±6.6335e-2i 5.9747e-1±5.5735e-2i 6.1570e-1 6.0650e-1]

Σ = [1.6128e+0 8.0119e-1 1.5063e-1 2.4812e-3 7.1005e-6 3.6561e-8 9.1467e-10 2.0197e-13]

Also the optimal eigenvalues for the case when $Q = I_8$ and $R = I_1$ are

[5.1085e-1±2.8557e-1i 5.9578e-1 6.1014e-1 5.1003e-1±6.0354e-2i 5.5294e-1±3.1867e-2i]

Again as in the continuous time case, it is desired to shape the open loop responses to the optimal ones. Partial pole placement will be used to achieve the design. A reduced order model of order $r = 4$ is first derived using discrete balancing [26], balagg and modal aggregation based on Davison. The eigenvalues of the reduced order model based on aggregation and balagg are

evar = [6.7322e-1±1.7791e-1i 6.1570e-1 6.0650e-1]

The reduced order model obtained via balancing has the following eigenvalues

evar = [6.7256e-1±1.7866e-1i 6.2003e-1±5.2442e-2i]

The reduced order eigenvalues are then placed to the dominant optimal eigenvalues which are

[5.5294e-1±3.1867e-2i 5.9578e-1 6.1014e-1]

The resulting closed loop systems are then obtained as in the continuous time case. The resulting closed loop eigenvalues based on aggregation and balagg are

$$evaf = [5.5294e-1 \pm 3.1867e-2i \quad 5.9578e-1 \quad 6.1014e-1$$
$$5.1456e-1 \pm 6.6335e-2i \quad 5.9747e-1 \pm 5.5735e-2i]$$

while those based on balancing are

$$evaf = [5.1498e-1 \pm 7.7573e-2i \quad 6.3384e-1 \pm 3.7204e-2i$$
$$5.5742e-1 \pm 6.4516e-2i \quad 4.9752e-1 \quad 6.0932e-1]$$

Figures (5.3.3-5.3.4) show the full order closed loop system responses when the placement is done based on the reduced order models. Also, the optimal response based on the full order system is also shown.

133

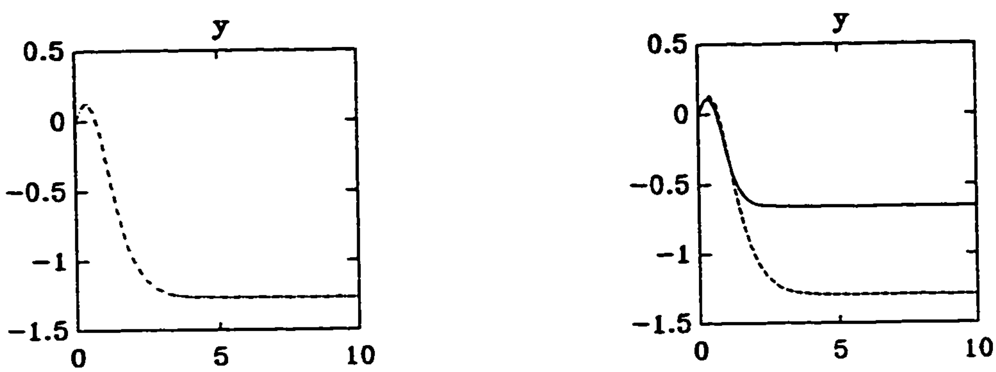


Fig.5.3.3 Closed Loop Responses. Pole Placement Based on:
(a) Balancing (–.); (b) Full Order System (–),
Aggregation (:), and Balagg (––) Based Systems; r = 4

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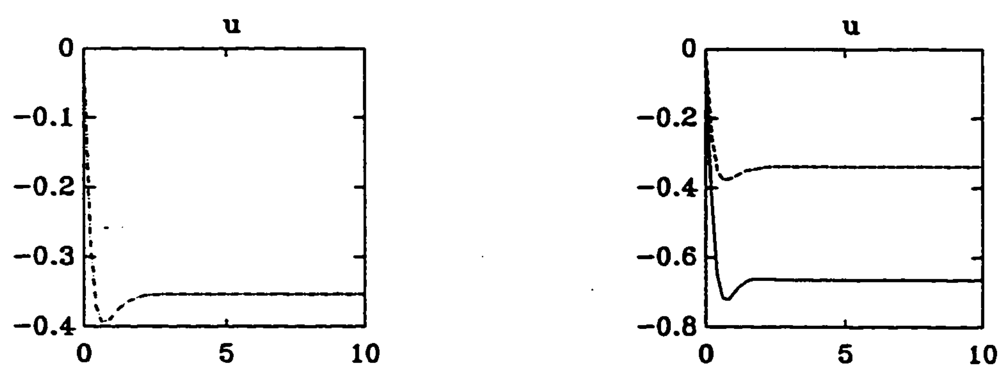

Fig.5.3.4 Closed Loop Controls. Pole Placement Based on:
(a) Balancing (–.); (b) Full Order System (–),
Aggregation (:), and Balagg (––) Based Systems; r = 4

5.4 Pole Placement Based on Reduced Order Models Based on Balancing, and Singular Perturbation: Comparison to Nominal Response

Pole placement based on reduced order models obtained by singular perturbation was discussed in chapter 3. It was stated that if it is desired to place the slow subsystem or the fast subsystem eigenvalues to λ_{si} , then the resulting closed loop eigenvalues will be at $[\lambda_{si}+O(\mu)]*1/\mu$. Moreover, it is possible that the closed loop eigenvalues become unstable.

In this section, shaping the closed loop responses by applying control laws based on reduced order models obtained by singular perturbation and balancing will be analyzed. This will be done for both continuous and discrete time systems.

5.4.1 Continuous Time Case: Voltage Regulator

Consider the voltage regulator given in section 4.7.1. It is desired to shape the open loop responses to the optimal ones when $R=I_1$ and $Q=c'c$ for different values of μ. Reduced order models of order $r=2$ are obtained via balancing and via singular perturbation. The poles of the reduced order models are placed to the dominant poles (when possible) of the optimal eigenvalues. Suboptimal control law is then derived from the reduced order model control law (by augmenting zeros) and then applied to the full order system. The results of the simulation are as follows:

1. For $\mu = 0.1$, both singular perturbation and balancing gave unstable closed loop systems.
2. For $\mu = 0.02$, singular perturbation gave unstable closed loop system but balancing resulted in a stable closed loop systems.
3. $\mu = 0.005$, both singular perturbation and balancing gave stable closed loop systems. In the following, only the results of this case will be given.

The open loop eigenvalues *eva* and the singular values of the balanced system Σ at $\mu = 0.005$ are:

eva = [-2.0000e-1 -5.0000e-1 -5.0000e+2 -2.8600e+2 -2.0000e+2]

Σ = [2.6325e+2 6.2549e+1 1.0378e+0 3.4190e-3 2.2681e-3]

The slow subsystem eigenvalues are

$evar$ = [-2.0000e-1 -5.0000e-1]

the eigenvalues of the reduced order model based on balancing are

$evar$ = [-1.9221e-1 -5.3722e-1]

the optimal eigenvalues are

[-5.8656e+2 ± 4.8720e+1i -3.3312e+1 -5.3065e-1 -2.1758e+2]

the dominant optimal eigenvalues are

[-3.3312e+1 -5.3065e-1]

the closed loop eigenvalues obtained via singular perturbation are

$evaf$ = [-4.4595e+2 -4.0705e+2 -6.6586e+1 ± 2.8544e+1i -5.3065e-1]

the closed loop eigenvalues obtained via balancing are

$evaf$ = [-5.1410e-1 -3.2960e+1 -4.9998e+2 -2.8626e+2 -2.0010e+2]

The responses and controls are shown in figures (5.4.1-5.4.2). From the figures, it can be seen that the responses of balancing and singular perturbation are almost identical. Also, the responses are almost of the same shape as the optimal ones except for a gain difference. The controls for all the situations are almost the same.

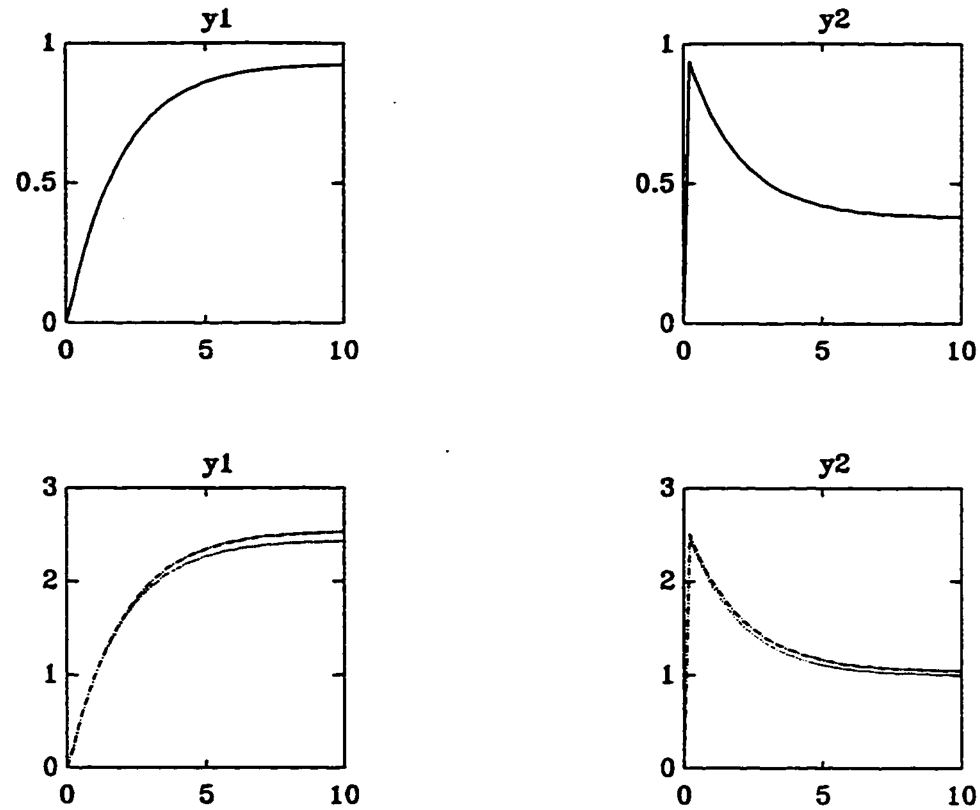

Fig.5.4.1 Closed Loop Responses. Pole Placement Based on:
(a) Full Order system (–); (b) Balancing (–.)
Singular Perturbation (:) Based Systems; u =.005

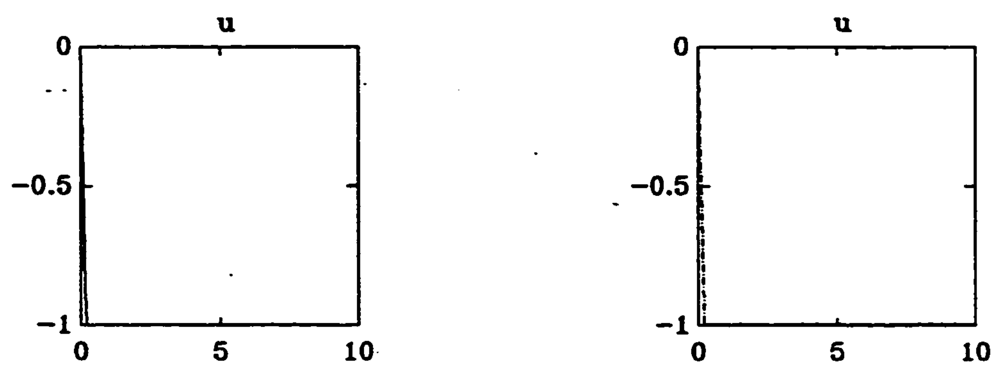

Fig.5.4.2 Closed Loop Controls. Pole Placement Based on:
(a) Full Order system (–); (b) Balancing (–.)
Singular Perturbation (:) Based Systems; u =.005

5.4.2 Discrete Time Case: Steam Power Plant

Consider the steam power plant given in section 4.8.1. It is desired to shape the open loop responses to the optimal ones when $R=I_1$ and $Q=c'c$ for $\mu=0.2646$. Reduced order models of order $r=2$ are obtained via balancing [26] and via singular perturbation. The poles of the reduced order models are placed to the dominant poles of the optimal eigenvalues. Suboptimal control law is then derived from the reduced order model control law (by augmenting zeros) and then applied to the full order system. The results of the simulation are as follows:

The open loop eigenvalues eva and the singular values of the balanced system Σ at $\mu=0.2646$ are:

eva = $[8.9974e-1 \pm 4.9958e-2i \;\; 7.9187e-3 \;\; 6.6172e-2 \pm 3.6990e-3i]$

Σ = $[9.5739e-1 \;\; 5.1593e-1 \;\; 1.716e-1 \;\; 2.0809e-2 \;\; 5.0833e-5]$

The slow subsystem eigenvalues are

$evar$ = $[7.8763e-1 \;\; 8.4928e-1]$

the eigenvalues of the reduced order model based on balancing are

$evar$ = $[5.5922e-1 \;\; 9.4589e-1]$

the optimal eigenvalues are

$[8.6990e-1 \;\; 8.3068e-1 \;\; 2.9857e-2 \;\; 1.8826e-1 \;\; 2.4012e-1]$

the dominant optimal eigenvalues are

$[8.6990e-1 \;\; 8.3068e-1]$

the closed loop eigenvalues obtained via singular perturbation are

$$evaf = [9.7379e-1 \;\; 7.6964e-1 \;\; 2.2913e-1 \;\; 2.8584e-1 \;\; 3.0018e-2]$$

the closed loop eigenvalues obtained via balancing are

$$evaf = [9.3844e-1 \pm 1.3251e-1i \;\; 3.0910e-2 \;\; 3.5011e-1 \;\; 2.5455e-1]$$

Figures (5.4.3-5.4.4) show the full order closed loop system responses when the placement is done based on the reduced order models. Also, the optimal response based on the full order system is also shown.

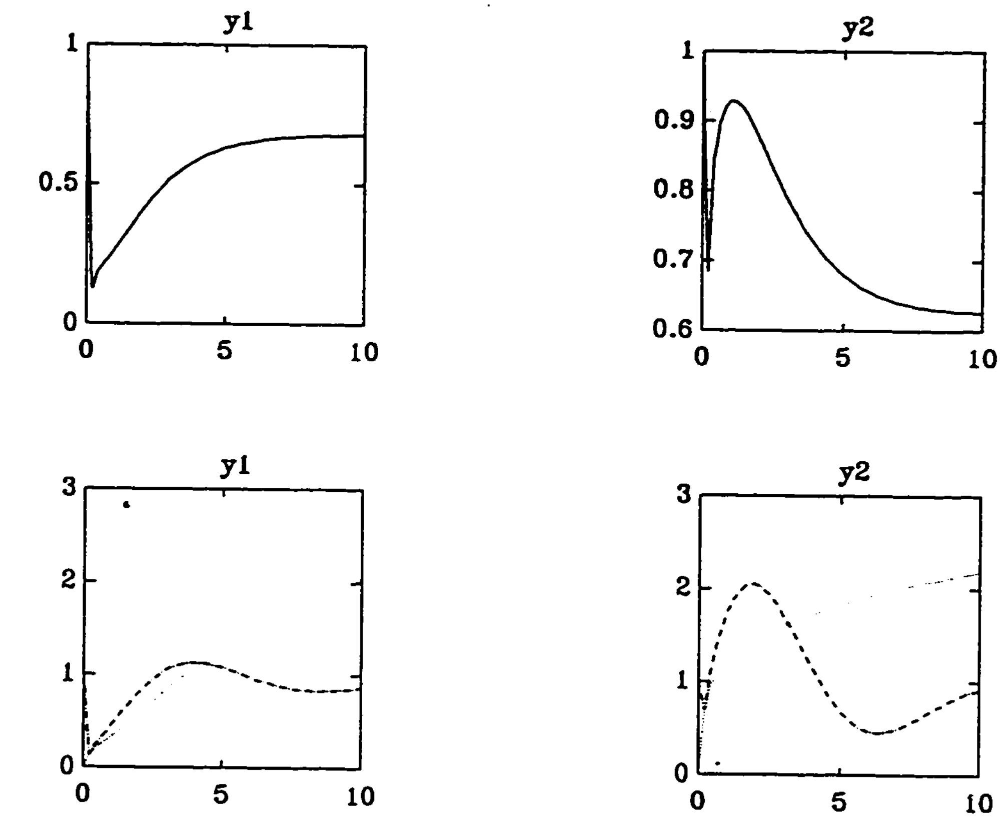

Fig.5.4.3 Closed Loop Responses. Pole Placement Based on:
(a) Full Order system (–); (b) Balancing (–.)
Singular Perturbation (:) Based Systems; u =.2646

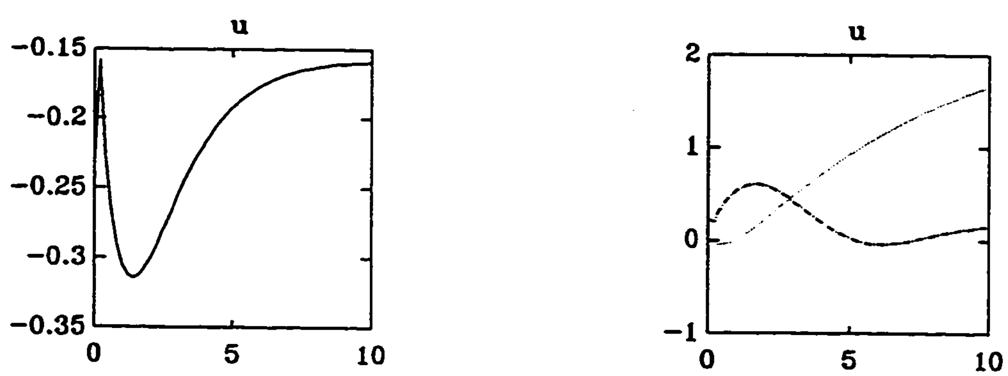

Fig.5.4.4 Closed Loop Controls. Pole Placement Based on:
(a) Full Order system (–); (b) Balancing (–.)
Singular Perturbation (:) Based Systems; u =.2646

5.5 Summary

Pole placement based on reduced order models has been considered. It was shown that if the purpose of the design is only to move some of the open loop eigenvalues to new locations and to leave the others unaffected, then pole placement based on reduced order models derived via aggregation and balagg will achieve the design purpose. However, pole placement based on reduced order models obtained via balancing does not always meet the design purpose. Also, pole placement based on reduced order models obtained via singular perturbation will never meet the design purpose. In fact the resulting closed loop systems based on balancing or singular perturbation can be unstable.

If the purpose of the pole placement is to shape the open loop response to a nominal curve like the optimal response, then non of the reduced order models derived via the different reduction methods will guarantee to give rise to closed loop responses close to the nominal ones.

In the next chapter, the overall conclusion of the thesis and suggestion for future work will be given.

133

CHAPTER 6

CONCLUSION AND SUGGESTIONS FOR FUTURE WORK

Model reduction has been of a great interest in control theory during the last three decades. Different reduction methods have been suggested. The concentration has been mainly on the behavior of the open loop reduced order models. The application of model reduction in suboptimal control has also been analyzed to some extent.

In this thesis, reduced order models derived using aggregation, singular perturbation, and balancing have been reviewed for continuous and discrete time cases. Suboptimal control based on reduced order models derived by aggregation and singular perturbation have also been reviewed. Although, balancing has been available for one decade, it has not been applied to suboptimal control design based on reduced order models. In this thesis, suboptimal control based on reduced order models obtained by balancing has been derived and evaluated for the LQR problem.

Also, pole placement based on reduced order models have been analyzed for continuous and discrete time cases.

6.1 LQR Suboptimal Control

The basic idea behind the proposed LQR suboptimal control law based on balancing is the fact that the LQR problem is affected the most by the most controllable and observable states of the system. Therefore, it is very logical to base the design of the suboptimal controller on the most controllable and observable part of the system.

The new LQR suboptimal control law based on balancing has been compared to suboptimal control laws based on aggregation and singular perturbation for both continuous and discrete time cases. The comparison was done through the application of the different suboptimal control laws derived by the different reduction methods to many practical systems. Suboptimal controllers of different orders have been obtained whenever possible. The basis of the comparison has been the optimal characteristics of the full order system. The simulation results can be summarized as follows:

1. **Stability:** Suboptimal control laws based on aggregation and balancing have always resulted in stable suboptimal closed loop systems. Theoretically, stability is guaranteed for suboptimal closed loop systems based on aggregation. Although, numerous practical systems based on balancing have been simulated and have given stable suboptimal closed loop systems, we have not been able to obtain theoretical results which

guarantee the stability of the suboptimal closed loop system based on balancing. Therefore, to overcome this problem, we have proposed a new reduction method, named Balagg, which utilized features of balancing and aggregation. The stability of the resulting suboptimal closed loop system is always guaranteed.

Suboptimal control laws based on singularly perturbed system do not always result in stable closed loop systems. The stability seems to depend on the value of μ which is a function of the ratio of the slow and fast time scales of the system.

2. **Suboptimality degree** ϵ**:** The suboptimality degree of suboptimal closed loop systems based on the different versions of balancing (i.e normal balancing [24][26] or 2-time scale balancing [35-37]) has shown to be very good and, in fact, in many cases it was very close to unity. However, when the suboptimal system is based on aggregation or singular perturbation, the suboptimality degree has been much lower than that of balancing. Also, suboptimality degree of systems based on balagg has been very low.

There is no relation that governs the behavior of the suboptimality degree ϵ as the order of the reduced order model is changed. However, it is noticed from the simulation that for closed loop systems based on balagg, the suboptimality degree increases as the order of the reduced order model becomes smaller.

3. Performance Degradation dJ**:** As in the suboptimality degree, the performance degradation has been the lowest for suboptimal closed loop systems based on balancing. For suboptimal systems based on aggregation, singular perturbation, and balagg, dJ has been much larger than that of balancing.

4. Suboptimal Step Responses and Controls: Again from the simulation, suboptimal responses and controls of suboptimal systems based on balancing have been extremely close to those of the optimal system. Responses and controls of suboptimal systems based on aggregation and singular perturbation have been, in general, far from optimum. When the suboptimal system is based on balagg, responses and controls have shown improvement over aggregation and became closer to the optimum ones. Almost, in all of the simulated cases, when the order of the reduced order model made smaller, suboptimal responses based on balagg did not deteriorate but, in fact, in some cases they became closer to the optimal ones.

5. Eigenvalue positions: In general, suboptimal eigenvalues are different from the optimal ones. However, in many cases they are close for suboptimal systems based on balancing.

133

6.2 Pole Placement

Pole placement based on reduced order models has been studied for two different design purposes. The first design purpose is to place only an r open loop eigenvalues while the remaining $(n-r)$ eigenvalues should not be altered. The second design purpose is to place all the open loop eigenvalues to nominal locations (which are chosen to be the optimal locations).

When reduced order models are used to achieve the first design purpose, it was seen that the only reduction methods that are capable of meeting the deign purpose are aggregation and balagg. Balancing does not always meet the design purpose and it can result in an unstable closed loop system. Singular perturbation never meets the design purpose and it can result in an unstable closed loop system.

When reduced order models are used to achieve the second design purpose, it was seen that non of the reduction methods is capable of meeting the deign purpose. All of them resulted in closed loop responses far from the optimal ones.

6.3 SUGGESTIONS FOR FUTURE WORK

1. The establishment of conditions under which the suboptimal control law based on balancing will result in a stable closed loop system.

133

2. Further analysis should be carried out towards the understanding of the new reduction method, balagg, and its behavior in closed loop.

3. The development of an algorithm to do singular value placement of balanced system.

133

Nomenclature

n: Full-order system order

r: Reduced-order system order

$S1$: Full-order continuous time system

$x(t)$: Full-order system states at time t

$u(t)$: Input at time t

$y(t)$: Full-order system output at time t

A: Full order system matrix

B: Full order input matrix

H: Full order output matrix

R^n: n-dimensional real space

$S2$: Reduced-order continuous time system

$x_1(t)$: Reduced-order system states at time t

$\hat{y}(t)$: Reduced-order system output at time t

F: Aggregated system matrix

G: Aggregated input matrix

D: Aggregated output matrix

C: Aggregation matrix

$(')$: Transpose

$e(t)$: Aggregation error

M: Modal matrix of $S1$

M_b: Modal matrix of the balanced system Sb

I_n: Identity matrix of order n

T^+: Pseudo-inverse of T

Λ: Modal matrix of Sl in Jordan form

Γ: Input matrix of Sl in Jordan form

λ_i: The i^{th} eigenvalue of A

Sd: Full-order discrete time system

Sdr: Reduced-order discrete time system

μ: Positive scalar representing the ratio of speeds of slow and fast subsystems

$\bar{x}(t)$: Quasi states

$\bar{y}(t)$: Quasi output

$x_s(t)$: Slow subsystem states

$u_s(t)$: Slow subsystem input

$y_s(t)$: Slow subsystem output

(A_o, B_o, C_o, D_o): Slow subsystem matrices

$x_f(t)$: Fast subsystem states

$u_f(t)$: Fast subsystem input

$y_f(t)$: Fast subsystem output

W_c: Controllability Grammian

W_o: Observability Grammian

T: Balancing transformation matrix

Σ: Singular values matrix of a balanced system

σ_i: i^{th} singular value of the balanced system

Sb: Balanced full order system

$x_b(t)$: Balanced full order system states

(A_b, B_b, C_b): Continuous time full order balanced system matrices

(Φ, Γ, E): Discrete time full order balanced system matrices

x_{bl}: Most controllable and observable states

x_{b2}: Least controllable and observable states

Σ_1: The matrix of the largest singular values

Σ_2: The matrix of the smallest singular values

L_c, L_o: Lower triangular Cholsky factor of W_c and W_o respectively

$tr(A)$: Trace of(A)

$\|\|_\infty$: Infinity norm

$\bar{\sigma}$: Maximum singular value

$\lambda_{max}(A), \lambda_{min}(A)$: Maximum and minimum eigenvalue of A respectively

(A_{11}, B_1, C_1): Reduced order continuous time model based on balancing

$(\Phi_{11}, \Gamma_1, E_1)$: Reduced order discrete time model based on balancing

J: Optimal cost functional

Q: States weighting matrix

R: Controls weighting matrix

u^*: Optimal Control

K: Optimal feedback gain

S^*: Optimal closed loop system

J^*: Optimal cost

J_m: Cost functional associated with the aggregated system

K_a: Optimal feedback gain for the aggregated system

K_{sa}: Suboptimal feedback gain for aggregation

J_s: Cost functional associated with the slow subsystem

u_s^*: Slow subsystem optimal control

K_{sp}: Slow subsystem optimal feedback gain

K_{ss}: Suboptimal feedback gain for singular perturbation

K_p: Feedback gain matrix for placing the aggregated system poles

K_{ps}: Feedback gain matrix for placing the slow subsystem poles

K_{pf}: Feedback gain matrix for placing the fast subsystem poles

J_{bm}: Cost functional associated with the reduced order model obtained from a balanced system.

Q_b: Weighting matrix in balanced coordinates.

Q_1: Weighting matrix for the reduced order system based on balancing.

K_b: Optimal Feedback gain matrix for the reduced order system based on balancing.

K_{sb}: Suboptimal Feedback gain matrix for balanced system.

ϵ: Suboptimallity degree

$\tilde{J}$: Suboptimal cost functional value

dJ: Performance degradation

eva: Open loop eigenvalues

$evar$: reduced order eigenvalues

K_{bp}: Feedback gain matrix for placing the reduced order eigenvalues obtained by balancing.

K_{bps}: Feedback gain matrix for pole placement based on balancing.

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133

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